MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

Vol. 96 No. 6

SPECIAL GENERAL MEETING

Professor H. C. Plummer, Vice-President, in the Chair.

A Special General Meeting of the Fellows was held at the Society's Apartments on Wednesday, the 8th of April, for the purpose of considering the following motion :—

"That notwithstanding Bye-law 41 the Ordinary Meeting of the Society in 1936 June be held on Wednesday, June 10."

The motion was formally proposed and seconded by the Secretaries on behalf of the Council, it being explained that the object of the proposed change of date was to enable those Fellows who were leaving London on June 12 for a Mediterranean cruise to view the total solar eclipse on June 19 to attend the June meeting of the Society.

The motion was carried by a majority.

MEETING OF 1936 APRIL 8

Professor H. C. Plummer, Vice-President, in the Chair.

The Chairman read the following letters which had been received in reply to the Loyal Addresses offered by the Society to H.M. the King and H.M. Queen Mary :—

HOME OFFICE,
WHITEHALL.
March 17th, 1936.

SIR,—I have had the honour to lay before The King the Loyal and Dutiful Address of the Council and Fellows of the Royal Astronomical Society on the occasion of the lamented death of His late Majesty King George the

Fifth, and have received The King's commands to convey to you His Majesty's grateful Thanks for the assurances of sympathy and devotion to which it gives expression.—I am, Sir, Your obedient Servant,

JOHN SIMON.

THE PRESIDENT,
Royal Astronomical Society.

HOME OFFICE,
WHITEHALL.
March 17th, 1936.

SIR,—I am directed by the Secretary of State to inform you that the Address of Condolence of the Council and Fellows of the Royal Astronomical Society on the death of His late Majesty King George the Fifth has been laid before Queen Mary, whose grateful Thanks I am to convey to you.—I am, Sir, Your obedient Servant,

R. R. SCOTT.

THE PRESIDENT,
Royal Astronomical Society.

The Chairman announced that the Council had invited Professor A. Kopff, of the Berlin Rechen-Institut, to deliver the George Darwin Lecture at the Ordinary Meeting of the Society in June.

The following candidate was proposed for election as a Fellow of the Society, the name of the proposer from personal knowledge being appended :—

Frank Mercer, Eywood Road, St. Albans, Herts (proposed by J. J. Hall).

The election by the Council of the following Fellows was duly confirmed :—

Eric Frank Baxter, M.A., M.Sc., Lecturer in Applied Mathematics, Durham Colleges in the University of Durham ; and

Mademoiselle Lucy Foltz, L'Observatoire, Digne, B.A., France.

Seventy-one presents were announced as having been received since the last meeting, including :—

Algiers, Observatoire : Catalogue fondamental de 872 étoiles rapporté à l'équinoxe moyen de 1925·0, par M. F. Gonnessiat ; and

Berlin-Babelsberg, Universitätssternwarte : R. Prager, Geschichte und Literatur des Lichtwechsels der veränderlichen Sterne, Band II, Cygnus-Ophiuchus.

The following Geophysical Meetings have been held during the present session :—

1935 Oct. 25. Geodesy in India. Opener, Capt. G. Bomford, R.E.

1935 Nov. 29. The Interpretation of Seismograms. Opener, Rev. J. P. Rowland, S.J.

1936 Jan. 31. (Joint meeting with the Royal Meteorological Society.) Ice Ages. Opener, Sir George Simpson, C.B., C.B.E., F.R.S.

1936 Feb. 28. Reading and Discussion of Papers presented to the Society.

1936 Apr. 24. Reading and Discussion of Papers presented to the Society.

THE ORIGIN OF THE SOLAR SYSTEM.

R. A. Lyttleton, B.A.

Introduction.—There is a general absence of precision in the theories which have been proposed from time to time in order to account for the peculiar dynamical arrangement of the bodies which constitute the solar system. A recent detailed survey of the whole question by Professor H. N. Russell in his treatise *The Solar System and its Origin* (Macmillan, New York, 1935) has emphasized the difficulties which these theories do not explain; and in particular the importance of the angular momentum criterion and the fatal consequence of its application to the present tidal theory are made perfectly clear for the first time. Briefly Professor Russell's objection is as follows:—

In the orbit of a star or any other body about the Sun the angular momentum per unit mass is proportional to $\sqrt{p(1+x)}$ where p is the semi-latus rectum and x is the ratio of the mass of the body to that of the Sun. Now to produce tidal action in the Sun the encounter must be fairly close and hence p cannot reasonably exceed 0·03 astronomical unit. If x is taken as unity it is then found that the angular momentum per unit mass of the passing star is 0·25 unit. Making a similar calculation for the planets the amounts vary from 0·61 unit for Mercury up to 5·49 units for Neptune. The weighted average for the planets is 2·63 units, which is ten times greater than the most favourable suppositions would allow them to have. So that, however attractive are the features of a system produced by a tidal or "side-swiping" encounter of the Sun with another star, such a theory is doomed from the start. (Jeffreys admits the grave nature of this criticism; *Nature*, 1935, p. 932.)

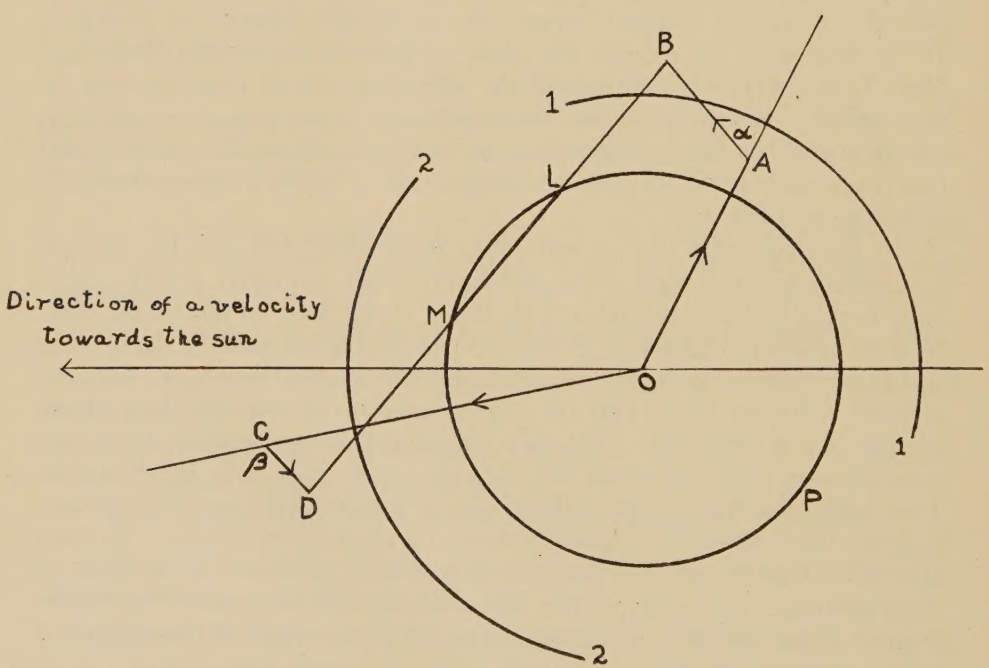
The suggestion that the Sun may have been a binary star is made by Professor Russell towards the end of his book in discussing a number of new lines of attack on the problem, but the difficulties of getting rid of the companion and producing planets are considered as more or less insuperable and the hypothesis is abandoned as unpromising. The same notion occurred to the writer independently, before finishing reading Professor Russell's book, in an endeavour to resolve the angular momentum difficulty. It then appeared that the difficulties were non-existent and that the very mechanism which could remove the companion of the Sun could also produce planets, namely, a tidal or "side-swiping" encounter of a third star with the companion.

1. To show that a binary system can be disrupted by means of the close approach to one component of another star of average mass and velocity (at infinity) and that all three stars can subsequently escape from each other:

An exact treatment of such a problem is impracticable and, moreover, unnecessary, for any information concerning the original system which might be sought in this way can indicate only orders of magnitude.

Let the Sun be denoted by M_0, its companion by M_1 and the intruder by M_2. Let a denote the distance M_0M_1. A very close encounter would be necessary to produce planets, and this is exactly the condition that during this encounter the Sun's action may be neglected and M_1 and M_2 regarded as describing hyperbolic orbits. (The validity of these approximations can be seen at once from the equations of motion referred to axes fixed in the Sun.)

In the following vector diagram an origin O is taken and the vectors $OA = U$ and $OC = V$ are drawn equal to the velocities of M_1 and M_2 just



before the hyperbolic approach begins. The circles 1 and 2 are drawn with radii

$$\sqrt{\frac{2(M_0+M_1)}{a}} \quad \text{and} \quad \sqrt{\frac{2(M_0+M_2)}{a}},$$

equal to the parabolic velocities of M_1 and M_2 respectively under the attraction of M_0 alone, so that A lies within the circle 1 and C outside 2. Also, in order to satisfy the requirements the velocities added by the encounter, viz. AB and CD, must take the ends of the velocity vectors of both M_1 and M_2 outside of their respective circles. Now if W is the relative

velocity of M_1 and M_2 just before encounter and e the eccentricity of their relative hyperbolic orbit, the velocities communicated to M_1 and M_2 are $2m_2W/e$ and $2m_1W/e$ in opposite directions where

$$m_1 = M_1/(M_1+M_2), \qquad m_2 = M_2/(M_1+M_2).$$

Suppose these velocities are added in directions making angles α and β with the original velocities (as shown) then writing $H = 2W/e$, the conditions of the problem require for M_1

$$\frac{M_0+M_1}{a} \leqslant U^2 < \frac{2(M_0+M_1)}{a}$$

$$U^2 + m_2^2H^2 + 2m_2UH\cos\alpha > \frac{2(M_0+M_1)}{a},$$

and for M_2

$$V^2 + m_1^2H^2 + 2m_1VH\cos\beta > \frac{2(M_0+M_2)}{a}.$$

For the purposes of a numerical example, let $M_2 = M_1 = M_0 = \odot$ and $a = 18$ astronomical units, then the unit of velocity is 30 km. per sec. and let $U = \sqrt{\dfrac{M_0+M_1}{a}} = \frac{1}{3}$. Considering the ejection of M_1, the most unfavourable case requires

$$m_2H > U + \sqrt{\frac{2(M_0+M_1)}{a}} > U(1+\sqrt{2}).$$

Now if d be the shortest distance between M_1 and M_2 during their encounter the eccentricity of the hyperbolic orbit is $1 + dW^2/(M_1+M_2)$. Hence the condition for the ejection of M_1 is

$$W > (1+\sqrt{2})\frac{U}{2m_2}\left(1 + \frac{dW^2}{M_1+M_2}\right).$$

If now $d = 1/40$ (a liberal estimate) the inequality becomes

$$W > \frac{1+\sqrt{2}}{3} + \frac{1+\sqrt{2}}{240}W^2$$

or

$$0{\cdot}8 < W < 99{\cdot}4.$$

Now

$$V^2 = V_\infty^2 + \frac{2(M_0+M_2)}{a} = V_\infty^2 + \frac{2}{9},$$

and since

$$V > U + 0{\cdot}8 = 1{\cdot}13,$$

this gives

$$V_\infty > 1{\cdot}0$$

approximately. The upper limit is of no special interest; a decrease in d sends it up rapidly and leaves the lower limit practically unaltered. So that in the worst case the star M_2 if properly projected at infinity need have

a velocity there of only about 30 km./sec. For the subsequent escape of M_2 the inequality is certainly satisfied if

$$m_1^2H^2 + 2m_1VH\cos\beta > 0$$

or

$$\cos\beta > -\frac{m_1H}{2V} = -\frac{W}{2eV} = -0{\cdot}35$$

approximately, and this is satisfied for a wide range of values of β, so that this simple approximate analysis of the encounter indicates that a close approach of the type suggested is fairly likely to effect the disruption of the binary system.

This discussion of such encounters has at least one other point of interest remote from the present subject. It is also possible that the intruding star M_2 would eject M_1 and in so doing be captured itself by M_0. This may perhaps account for the existence of anomalous binary systems, *e.g.* Sirius and its famous companion, for this makes it possible to avoid the formidable puzzle as to how two such different objects could have evolved from originally similar matter.

This elegant vectorial treatment of the disruption was suggested by Professor Russell, and it is a pleasure to record my thanks to him for dispensing with the inconvenient algebra which gave these results in the first place.

2. During the encounter which ejected the Sun's companion the planets are supposed to have been formed. The most plausible theories of the formation of planets by an encounter agree in supposing that a long filament is produced between the two stars M_1 and M_2 either by tidal action or a grazing collision or a combination of both processes. In this paper it is assumed that some such filament developed between the Sun's companion and the intruder and that the filament subsequently condensed and the primitive planets formed.

It is next shown that if a series of planets were formed in this way, then it is extremely likely that some of them would be captured by the Sun and, furthermore, might all proceed round the Sun in the same general direction.

In the vector diagram the points B and D are the ends of the velocity vectors of M_1 and M_2 after the encounter is over. So that if the velocities of the condensations in the filament were distributed more or less uniformly between the velocities of the two stars the ends of the velocity vectors of the primitive planets would lie on or near BD and let this be called "the line of planets." If a planet is to be captured by the Sun its velocity must be less than the parabolic velocity of a small body at the distance a, and hence planets will be captured if the line of planets cuts the circle P whose radius is this parabolic velocity, in real points. And it is also seen that the velocities given by points on such a chord as LM, though differing greatly in the amount of angular momentum per unit mass, can easily be in the same direction round the Sun.

The condition that the radius p of P is large enough for some of the filament to be captured can easily be found as follows : if M_1 and M_2 escape

with velocities U_1 and V_1 in directions making an angle θ with each other and we define θ_1 and θ_2 by

$$U_1 \cos \theta_1 = p = V_1 \cos \theta_2$$

then the condition is $\theta > \theta_1 + \theta_2$.

If M_1 and M_2 just escape in directions at right angles this gives $M_0^2 > M_1 M_2$.

In the preceding discussion the motion is regarded as taking place in one plane, but if the circles in the vector diagram are replaced by spheres the whole of the argument is still valid, so that the possibility of such an encounter disrupting the system and leaving behind planets is clearly demonstrated.

The greatest angular momentum per unit mass which could be given to a planet corresponds to the point L being at its greatest distance from the line drawn toward the Sun, and the amount is accordingly $\sqrt{2M_0 a}$, so that a planetary system of circular orbits produced from a binary star in the manner suggested cannot have more than twice the range of the orbit of the binary. If Pluto be the outermost member of the solar system, this result indicates that the Sun's companion would have been at least 20 astronomical units from the Sun, a value which agrees well with the separations of visual binaries whose spectral class is similar to that of the Sun.

The spins of the planets present some difficulty. On the collision or side-swiping theory of the encounter the rotations are accounted for naturally as the result of the viscous forces during the grazing collision. If, however, the planets were produced solely by tidal action the spins can be attributed to the angular momentum of the matter when it formed part of either star, though it is not known how rapidly these may have been rotating. Other considerations have shown that it is necessary to consider the rotations of the planets as having been primitive, and this assumption is adopted here. Perhaps the high inclination of the axis of Uranus can be identified (more or less) with the axis of rotation of the intruding star.

3. *The Satellites.*—The initial velocities of the planets will lie nearly in a plane, but unless this plane passes through the Sun (approximately) the orbits of the primitive planetary system will not be of small mutual inclinations. Since more than half the angular momentum of the solar system is contributed by Jupiter it is fair to surmise that the plane of Jupiter's orbit has never been very different from the present one, and hence this marks the plane through the Sun and some point of LM (perhaps about the mid-point of BD). Now in any case the relative inclinations can at most be about equal to the angle LOM, and if this were small the distribution of angular momentum per unit mass would not vary much from planet to planet. Hence it seems probable that the plane of the orbit of the hyperbolic encounter was nearly that of the plane of the original binary. Dynamical investigation might show that such an assumption is unnecessary, but in any case the assumption is not essential to the following suggestions concerning the subsequent formation of satellites.

The primitive planets, while condensing out of the filament, were in a

comparatively small region where their mutual perturbations were considerable, but they could certainly escape each other, for otherwise they could not have condensed into separate bodies. As soon as the planets got outside this region they began to describe elliptical orbits (to a high approximation), and hence each planet must return to this region, unless it received considerable perturbation elsewhere in its orbit, which occurrence, it will be seen later, is not any real objection to the present argument.

It is here suggested that the subsequent close approaches of pairs of liquid planets are responsible for the production of the satellites.

Consider two orbital paths not necessarily in the same plane which pass near each other. It is clear that in a general way the faster moving planet is equally likely to pass the other in any plane through the centre of the latter, and accordingly it would seem that if satellites were formed in this way they would be equally likely to be direct or retrograde or of high inclinations. Now this is by no means the case with actual satellites, and the following discussion appears to indicate the reason. Consider a planet M and a disturbing planet M' which passes with (relative) velocity v at a distance R from M in its equatorial plane and in the same sense as the rotation of the planet M. The relative motions of the different parts of M being in question it is reasonable to adopt axes rotating with the angular velocity ω of this planet, and referred to these axes the velocity of the passing planet is $v - R\omega$. Similarly if the disturbing planet pass M in the opposite sense to its rotation the velocity referred to axes fixed in M is $v + R\omega$. (A small inclination of the direction of v to the equatorial plane will affect these values but slightly.)

Now Jeans has shown (*Problems of Cosmogony and Stellar Dynamics*, pp. 46 and 130, Cambridge University Press, 1919) that for a slow encounter the closest distance R of safe approach of a point mass M' to a homogeneous gravitating mass M is given by

$$R = 2{\cdot}2\left(\frac{M'}{M}\right)^{\frac{1}{3}} r_0, \tag{i}$$

where r_0 is the undisturbed radius of M. (If the centrifugal force were not neglected the value of R must be greater and for a circular orbit of M' about M the constant factor becomes 2·46.) Also for a transitory or rapid encounter the closest distance is given by (using astronomical units of mass and length)

$$R^2 = 3{\cdot}03\frac{M' r_0^{\frac{3}{2}}}{v M^{\frac{1}{2}}}. \tag{ii}$$

If it can be shown that $v - R\omega$ becomes small for a pair of planets, say Earth and Jupiter, at the distance R at which Jupiter would endanger the stability of the Earth in a slow encounter, then the use of formula (i) for finding this distance is justified. For encounters in which M' passes "against the spin" formula (ii) can be used.

Consider the case of an encounter between the Earth and Jupiter.

For the instability of the Earth the slow encounter formula requires Jupiter to approach to within

$$2{\cdot}2\ (M_J/M_E)^{\frac{1}{3}} \text{ times the radius of Earth} = 15r_E = 60{,}000 \text{ miles.}$$

For the disturbance of Jupiter by the Earth the distance is

$$2{\cdot}2\ (M_E/M_J)^{\frac{1}{3}} \text{ times the radius of Jupiter} = 0{\cdot}3r_J = 15{,}000 \text{ miles.}$$

Hence if Jupiter disturb the Earth tidally at a distance of 60,000 miles it will itself remain practically undisturbed by the Earth. The relative velocity during such an encounter is given by

$$v^2 = \frac{2}{15}\frac{M_E + M_J}{r_E} + v_0^2,$$

and taking v_0 to be about 10 km. per sec. we find $v = 52$ km./sec. The value of $R\omega$ for the Earth (assuming 4 hours for the initial period of the Earth's rotation) is found to be 42 km./sec. Hence

$$v - R\omega = 10 \text{ km. per sec.},$$
$$v + R\omega = 94 \text{ km. per sec.}$$

The direct encounter is therefore slow while the retrograde or indirect encounter is rapid; also encounters in meridional planes are rapid. Using formula (ii) for the indirect encounter the distance comes to 30,000 miles, and this being less than the radius of Jupiter it shows that tidal action during a retrograde encounter would not produce a satellite. Similarly no satellite could be produced by an encounter in a meridional plane. It is also worth remarking that the angular momentum of the Moon per unit mass is only a small fraction of that of Jupiter in such an encounter as is here advocated.

Since Mars has slow rotation all encounters will be rapid for it, and accordingly satellites are unlikely, but with a slight preference in favour of direct satellites. Again the angular momentum criterion is satisfied.

Next consider the action of Jupiter on Neptune. For a slow encounter to have any suitable tidal effect requires

$$R = 2{\cdot}2(M_J/M_N)^{\frac{1}{3}}r_N = 5{\cdot}6r_N = 92{,}000 \text{ miles.}$$

For Neptune to endanger Jupiter requires

$$R = 2{\cdot}2(M_N/M_J)^{\frac{1}{3}}r_J = 0{\cdot}83r_J,$$

so that Neptune can scarcely tidally affect Jupiter.

Now

$$R\omega \text{ at 5 radii for Neptune} = 16 \text{ km. per sec.},$$

and

$$\text{the relative orbital velocity} = 70 \text{ km. per sec.}$$

Hence any encounter between Neptune and Jupiter is transitory, and accordingly R will have to be less than 90,000 miles.

Suppose, therefore, Jupiter taken as near Neptune as is practicable, say just over 60,000 miles, then

$$R\omega = 10 \text{ km. per sec.},$$
$$\text{Orbital velocity} = 90 \text{ km. per sec.}$$

Hence $v + R\omega = 100$ kms. per sec. $= 3\frac{1}{3}$ units of velocity and formula (ii) then gives $R = 50{,}000$ miles.

Similarly $v - R\omega = 80$ requires $R = 56{,}000$ miles.

The value $R = 50{,}000$ indicates a grazing collision, but these formulæ not being accurate it is not possible to say definitely whether a collision is essential to account for a satellite of Neptune. However, they do show Neptune is almost equally unlikely to have either direct or retrograde satellites on a tidal theory. The angular momentum per unit mass of the satellite of Neptune is at present only a fraction of that which would be available during even a grazing collision.

The action of Jupiter on the rapidly rotating Uranus can readily produce direct satellites on this theory.

It is not easy to see how satellites could be produced out of Jupiter itself by tidal action, for even Saturn could not quite get near enough. However, Jupiter could produce direct satellites for Saturn by a direct encounter within about 125,000 miles (between centres), and it is possible that some of these satellites could be captured by Jupiter. It may be remarked that such satellites would have direct motion round Jupiter.

The angular momentum per unit mass in the case of the satellites of Saturn and Jupiter is of the same order of magnitude as is available during a grazing encounter. Taking the mass of the primary as unit and using the astronomical unit of length, the angular momentum per unit mass of Jupiter in an encounter to within 80,000 miles of Saturn comes out to be 0·1 unit.

For	Mimas the value is	0·035,
	Titan	0·090,
and	Iapetus	0·154.

Similarly for the motion of Saturn about Jupiter the angular momentum per unit mass is 0·047 unit.

For	Io the value is	0·053,
	Callisto	0·112,
	VII	0·280.

Thus the difficulty with regard to angular momentum occurs again for these satellites, but the discrepancy is here nothing like so large as in the theory of the production of planets from the Sun. It is possible that during partial solidification of the fast spinning planets Jupiter and Saturn, tidal friction may have been very great, and though existing for a comparatively short time could produce a considerable effect; so that the angular momentum difficulty is not serious in this case. The outer satellites of Jupiter and Saturn seem most readily and satisfactorily explained as captures at probably a much later stage.

4. Each of these encounters may produce an arm of satellites just as in the case of the original filament of planets, and some of the members may escape from both planets, but they cannot escape the Sun's gravitation and may therefore remain to form asteroids.

A rough calculation shows that encounters as close as these would occur only at rare intervals, probably of the order of millions of years; but this is a theoretical advantage, for the satellite systems seem more the result of single encounters than the surviving effects of many. Each of a series of close encounters might cause the removal of some or all of the previously existing satellites, thereby increasing the number of asteroids, and produce a new satellite system, in which case the existing states represent the effect of the last encounters. These assumptions require that throughout the period of the encounters the planets were either fluid or else subject to disruptive forces great enough to break up solid bodies. For the major planets this presents no serious difficulty, for it is even now uncertain what portions of their masses are still liquid or gaseous.

To account for the Moon it must be assumed that an encounter occurred while a considerable part of the silicate layer of the Earth was still fluid. Jeffreys' estimate that solidification would begin after about 15,000 years depends on the assumption that the mass would solidify first at the top; but even so the crust so formed would have to be fairly thick to avoid disruption by the forces here suggested. His later conclusion that solidification began at the bottom of the silicate layer and progressed gradually upwards leaves a much longer interval available. Further, Jeffreys has presented insuperable objections to the hypothesis of the origin of the Moon by tidal resonance: the present hypothesis offers at least what appears to be a possible explanation of the existence of the Moon since its small mass makes it doubtful whether it could have been formed directly by the encounter which produced the planets.

Again, the encounters might assist greatly towards the rounding up of the orbits of the planets, but could not alone completely effect this; indeed, an encounter between Neptune and Pluto is still possible. But this raises a difficult question, and we can always fall back on the resisting medium, and the encounter which produced the planets could, and probably would, also have produced the resisting medium, and, moreover, in a region remote from the Sun which is both where it is required and where it is most effective.

If the interactions between the planets are sufficiently close to produce satellites it is quite possible that a local resisting medium might be produced besides, and this medium would be able to round up the satellite orbits.

Also in the case of Jupiter and Saturn it is likely that they remained almost entirely gaseous for a considerable time, so that satellites produced from them, by a very close encounter or a grazing collision if necessary, would be of low mean density. These few remarks concerning the interactions of the planets are only tentative and probably by no means exhaust the possibilities.

Summary.—(i) The hypothesis of the Sun having been a double star

is adopted and the facility with which the companion can be removed is demonstrated.

(ii) The mechanism which effects this disruption also produces the planets, giving them widely varying angular momentum per unit mass and possibly causing them all to proceed round the Sun in the same general direction nearly in a plane.

(iii) The later encounters between pairs of planets while in the liquid stage give promise of an explanation of satellites in the equatorial planes of the planets and may even be able to account for retrograde satellites.

In conclusion I take this opportunity of expressing my deep indebtedness to Professor H. N. Russell for his continued advice during the preparation of this paper and the patience he has shown in elucidating for me obscure points of the literature relating to this subject.

Princeton University Observatory :
1936 *March* 17.

ON THE *K* TERM OF THE RADIAL VELOCITIES OF B-TYPE STARS.

W. M. Smart, M.A., D.Sc.

(Communicated by the Director, Cambridge Observatory)

1. In a recent paper * (which we shall afterwards refer to as SG) by Mr. H. E. Green and the author, we arrived at the value of $+4{\cdot}8$ km./sec. for the K term for stars of type Bo–B9. The material consisted of 630 radial velocities taken from Schlesinger's *Catalogue of Bright Stars*, 1930 (limiting magnitude $6^{m}{\cdot}5$), and the solutions took account of the galactic rotational terms. Actually it was found that the inclusion of the rotational terms made little difference to the value of K as deduced from a straightforward solution for the solar motion alone. We drew attention to the fact that our value of K, of the same order of magnitude as found by previous investigators, differed from the value given by Dr. J. S. Plaskett and Dr. J. A. Pearce in a recent paper † (which I shall subsequently refer to as PP), and we suggested that their mode of treatment of some of the observational material was incorrect. In the present communication I propose to deal with the criticism in greater detail and I hope to demonstrate the weakness of their claim (PP, 691) that "the famous K term in the Class B stars, whose high positive value of nearly $+5$ km./sec. has hitherto successfully evaded explanation, has now been reduced, by taking into account all known systematic motions, to $+1{\cdot}1$ km./sec., and this remaining displacement is practically the same as the computed Einstein red-shift of the spectral lines in these stars." The subject is an important one, and no apology is required for a thorough and frank discussion.

* *M.N.*, 96, 471, 1936 † *M.N.*, 94, 679, 1934.

2. The criticism arises in connection with what Plaskett and Pearce call the "stars of the southern stream." These are stars between galactic longitudes 210° and 330° and mostly close to the galactic equator. Following up the results of an earlier paper by Kapteyn,* Plaskett and Pearce re-investigated the proper motions of the stars in the longitude range concerned and suggested that the transverse motions of 154 stars led to the kind of convergency long familiar in the case of the Taurus Stream. The convergent point was determined to be (194°, −24°) in galactic co-ordinates, which is about 8° away from the position of the solar antapex as quoted by Plaskett and Pearce. It may be noted in passing that, doubtless owing to the smallness of the proper motions, the test of convergency was not very stringent; the stars were divided into eight groups, and the maximum deviation from the mean value of a group which they allowed was 70°; there were 4 stars with deviations between 50° and 70°, 17 with deviations between 30° and 50°, and the average deviation for all the stars was 15°. Such deviations belong to a different category from that associated with such a stream as the Taurus Cluster. Of the 154 stars, the radial velocities of 132 were subsequently used.

The relevant details (PP, 684) which we shall consider are in Table I.

TABLE I

Data of the "Southern Stream"

Group	Limits (Galactic Co-ordinates)	λ	Proper Motions		Radial Velocities	
			No.	$\bar{\mu}$	No.	$\bar{V}$ (km./sec.)
1	210°–240° ; −15° to + 8°	33·4°	25	0·0253″	20	+17·8
2	240°–260° ; −31° to +20°	53·1°	19	229	18	+18·3
3	260°–280° ; −20° to + 2°	73·2°	17	361	15	+12·9
4	260°–280° ; + 2° to +24°	80·5°	19	376	15	+14·4
5	280°–305° ; −20° to +10°	98·6°	15	291	13	+ 6·3
6	280°–305° ; +10° to +30°	102·7°	24	372	23	+ 5·7
7	305°–330° ; +10° to +30°	126·7°	24	308	18	− 2·5
8	305°–330° ; −18° to +10°	117·2°	11	357	10	+ 1·0

In this table, λ denotes the angular distance of the centre of each group from the convergent point (Plaskett and Pearce use the angular distance, D, from the point of the celestial sphere antipodal to the convergent point; consequently, the "stream velocity" is found with a negative sign; also, $\lambda = 180° - D$).

3. Consider the radial velocities. If the stars concerned form a "stream," with a common velocity, V_0, relative to the Sun and directed towards the convergent point, we have the following relation between V_0, λ and $\bar{V}$ for a given group :—

$$V_0 \cos \lambda + K = \bar{V}. \qquad (1)$$

* *Ap. J.*, 40, 43, 1914.

This formula is merely a simplified form of the usual equation for finding the solar motion from radial velocities ; in (1) we are not required to find the direction of V_0 as this direction is supplied by the proper motions ; moreover, the stars to which the equation is applied have, by hypothesis, no systematic motion relative to their centroid, so that the equation is theoretically an accurate one.

In (1), K is the K term as ordinarily understood. It includes the systematical gravitation red-shift which, according to the calculations (PP, 691) of Plaskett and Pearce, amounts to $+2{\cdot}6$, $+1{\cdot}7$, $+1{\cdot}0$ km./sec. for the O, Bo–B2 and B3–B5 stars respectively. Accordingly, if we insert a K term in the equation of condition (1), it must be in the sense that K is a purely systematic quantity whose existence *may* be explained partly by the gravitational displacement of the spectral lines and partly, perhaps, by a systematic error in the adopted wave-lengths used in the measurement of the radial velocities of these stars. But Plaskett and Pearce give K a kinematical interpretation and define it as "the average outward radial component of the stream" (PP, 685). But the average outward radial velocity for the centre of a given region is $V_0 \cos \lambda$, if we assume our stars to form a stream ; consequently Plaskett and Pearce's interpretation of K is erroneous.

When (1) is solved by least squares for the eight groups, the results are

$$\begin{aligned} V_0 &= +15{\cdot}0 \pm 1{\cdot}0 \text{ (km./sec.)} \\ K &= +\ 8{\cdot}3 \pm 0{\cdot}5 \text{ (} \quad ,, \quad \text{)} \end{aligned} \tag{2}$$

The interpretation of these results is simply that the solar motion *with respect to the group of stars* (132 in number) is 15·0 km./sec. directed very nearly towards the accepted position of the solar apex and *that the K term comes out unexpectedly large.*

4. However, having determined the values of V_0 and K in (2), Plaskett and Pearce proceed to write (1) as

$$K = \overline{V} - V_0 \cos \lambda, \tag{3}$$

and from the tabular values of $\overline{V}$, λ and the value of V_0 in (2) they "calculate" the value of K for each group (they denote it by $\bar{K}$). The value of each $\bar{K}$ is, however, the solution value $+8{\cdot}3$ together with the *residual* for the given group. They state (PP, 686) that the value of the "radial component of the stream" of $+8{\cdot}3$ km./sec. is the average over all the groups, but owing to the fact that the material is collected from a range of galactic longitudes amounting to 120°, the "radial component" will be larger at the middle and smaller at the ends of this longitude interval. They then go on to say that it is preferable to use the smoothed values of $\bar{K}$ "*which will obviously be proportional to sin* λ and which must be adjusted so that the average K for the whole stream is $+8{\cdot}3$ km./sec." These arguments are clearly erroneous. The equations of condition (1) are solved on the understanding that K is a constant ; if their subsequent arguments were correct, it follows that (1) should have been written as

$$V_0 \cos \lambda + K \sin \lambda = \overline{V},$$

an equation that has no kinematical or physical authority. Moreover, their argument that $\bar{K}$ is systematic (proportional to sin λ) implies that the *residuals*, previously mentioned, are also systematic. But as they find the probable error of V_0 and K, this is contrary to the implied assumption that the residuals are accidental in character.

5. It is now easy to see how they succeed in reducing the K term for all the stars—849 in number—which they utilise for the general solution with galactic rotational terms. For, believing that K signifies an outward radial velocity for the 132 stars of the "southern stream," they remove the "smoothed" values $\bar{K}$ for the stars in the eight groups of Table I from the observed radial velocities. Roughly, the effect of this operation produces a "reduction" of the K term by

$$\frac{132 \times 8{\cdot}3}{849} \quad \text{or} \quad 1{\cdot}3 \text{ km./sec.},$$

assuming that each radial velocity carries equal weight. Actually, the "reduction" is stated to be 2·0 km./sec. (PP, Table IV, lines 7 and 8, 689)—the orthodox solution for the 849 stars gives $K = +3{\cdot}1$ km./sec. and the general solution, in which the 132 stars of the "southern stream" are treated by the method described, gives $K = 1{\cdot}1$ km./sec., which Plaskett and Pearce calculate to be of the order of magnitude of the gravitational red-shift of the spectral lines. It may be noted that in the orthodox solution (PP, Table I, 683) the values of K calculated from seven groups of stars (these groups are virtually formed according to distance) vary between $+5{\cdot}9$ to $-1{\cdot}5$ km./sec., omitting two groups of relatively small weight.

6. If there were any outstanding anomalies in the radial velocities of the B-type stars, they might be expected to be revealed in the statistics used by Mr. Green and myself (SG).

Table II gives our regions with the number of stars and the average algebraic residual,* r, obtained from the equation

$$lX + mY + nZ + K + pu + qv - \bar{\rho} = r \qquad (4)$$

in which (l, m, n) are the direction cosines of the centre of a region with galactic co-ordinates (G, g), $(-X, -Y, -Z)$ are the components of the solar velocity, $\bar{\rho}$ is the mean observed radial velocity, and

$$p = \sin 2G \cos^2 g, \qquad q = -\cos 2G \cos^2 g,$$
$$u = \bar{r}A \cos 2G_1, \qquad v = \bar{r}A \sin 2G_1,$$

in which G_1 is the longitude of the centre of the galaxy. The values of X, Y, Z, K, u and v used to calculate r from (4) are those obtained for all B-type stars within 60° of the galactic equator (SG, Table VIII, column 4).

The regions † in Table II marked with an asterisk are within the limits of longitude defining the boundaries of Plaskett and Pearce's "southern stream." Our region K includes approximately groups 1, 2 and 3 of Table I, and region l includes groups 6 and 7.

* I am indebted to Mr. Green for the calculations.

† For details of the regions, see *M.N.*, **96**, 168, 1936.

TABLE II

Region	No. of Stars	Average Residual (km./sec.)	Region	No. of Stars	Average Residual (km./sec.)
*h**	30	+2·33	*H*	34	+2·71
i	13	−0·77	*I*	38	0·00
j	30	−3·71	*J*	41	+1·93
k	43	0·00	*K**	52	−2·50
*e**	13	−0·39	*E*	7	+1·00
f	11	+2·91	*F*	8	−3·75
g	21	+0·95	*G**	9	+8·44
*l**	45	−3·18	*L*	25	−1·92
m	12	+1·33	*M*	65	−1·38
n	16	+2·94	*N**	21	−1·20
b	6	−1·33	*B*	4	−9·50
c	12	+4·92	*C*	5	−5·80
d	11	−4·64	*D*	7	+4·00
p	7	+8·71	*P*	13	−2·38
q	6	+8·50	*Q*	14	+7·93
r	7	−2·00	*R*	4	+8·00

Excluding the third group of regions (*b* to *R*) centred at galactic latitudes $\pm 45°$, which include few stars, the average residual per group is about 2 km./sec. numerically and the regions with asterisks are no more remarkable than the others. (It is true, of course, that *G* has a large positive residual, but ignoring the fact that the number of stars in this region is 9 only, its sign is opposite to the signs in *K* and *l* where the anomalies, if they exist, should be shown.)

I conclude that the radial velocities of over 630 stars are as normal as can be expected and that they give no indication of pronounced "streaming" in the suspected range of longitude.

7. I interpolate here results which were omitted from the paper by Mr. Green and myself (SG). We have found that the mean speed of the B-type stars is 5·8 km./sec. (freed from solar motion and galactic rotation) and 6·0 km./sec. from the equations without the rotational terms.

Campbell and Moore's values * from 284 stars of types Oe5 – B5 and 500 stars of types B8 – A3 are 8·7 and 9·9 km./sec. respectively.

Our results appear to show that the system of B-type stars is even less mobile than was previously believed.

8. We now consider the mean proper motions, $\bar{\mu}$, in the eight groups of Table I. We assume that the stars concerned form a stream. Let *M* denote the proper motion at a point 90° distant from the convergent point. Then, assuming for the moment that all the proper motions are directed exactly towards the convergent point, we have, for each group, the equation

$$M \sin \lambda = \bar{\mu}_0, \tag{5}$$

where μ_0 denotes the proper motion towards the convergent point. As

* *Lick Publ.*, **16**, 38, 1928.

already mentioned, the average deviation from convergency is 15°. If θ denotes the deviation of a single star, $\mu_0 = \mu \cos \theta$ and $\bar{\mu}_0$ is then the mean value of $(\mu \cos \theta)$ for all the stars in a group. Thus $\bar{\mu}_0$ will be slightly less than $\bar{\mu}$. As Plaskett and Pearce have not yet published the proper motion data for the individual stars, the value of $\bar{\mu}_0$ cannot be found exactly.

The equations (5) have been solved for M, using the values $\bar{\mu}$ as given in Table I in place of the unknown values $\bar{\mu}_0$; the result is

$$M = 0''.0365 \pm 0''.0025. \qquad (6)$$

If $\bar{p}$ is the mean parallax of the stars,

$$4.74\, M = \bar{p} V_0,$$

from which, inserting the value of M,

$$\bar{p} V_0 = 0.173. \qquad (7)$$

The mean parallaxes of the stars concerned, in several subdivisions as regards spectral type and magnitude, have been estimated by Plaskett and Pearce (PP, 701). From an inspection of their values, $\bar{p}$ is unlikely to exceed 0″.005. The mean distance of the B-type stars brighter than $6^m.5$ has been found by Mr. Green and myself to be 170 parsecs (SG, Table VIII). Taking this as the minimum distance we find, from (7), for the minimum value of V_0,

$$V_0 = 29.4 \text{ km./sec.},$$

which is just about twice the value derived from the radial velocities by means of (1).

The two methods of determining V_0 thus lead to very different results. The mean is 22.2 km./sec.—if we are justified in taking the mean of two such discrepant values—and this is practically the value of the solar motion (21.5 km./sec.) in SG, Table VIII.

It is highly probable that the proper motions carry relatively large errors, making the values of $\bar{\mu}$ in Table I too large. This is borne out by the large value of M, which is nearly equal to that (0″.0385) obtained by L. Boss * from the proper motions—of all spectral types—in the *Preliminary General Catalogue*. With Boss's value of M and the normal value of the solar motion the mean parallax of the Boss stars is 0″.009, and this is certainly larger than the mean parallax of the B-type stars for the same range of magnitudes.

9. *Conclusion.*—If the preceding arguments are accepted it must be admitted that, according to all the evidence from radial velocities, the K term for the B-type stars has a value between +3 and +5 km./sec., a value much greater than the estimated gravitational displacement. We are thus left with the problem, still to be solved, of elucidating the kinematical or physical significance of the K term.

It may also be asked: What is the significance of the "Southern Stream"? As stated in section 7, the mean speed of the B-type stars (freed from the

* *A.J.*, 26, 112, 1910.

solar motion and the rotational effects) is small, and the consequence of this is that the B-type stars with their small *motus peculiares* can display the solar motion more clearly than any other type. In other words, a considerable proportion of B-type stars within 90° of the antapex ought to show an apparent cluster motion towards the antapex, and a like proportion of stars within 90° of the apex ought to show an equally pronounced cluster motion away from the apex. The "Southern Stream," it would appear, consists of those *selected* stars which show this kind of convergency most favourably.

Observatory, Cambridge :
1936 *March* 31.

NOTE ON RADIATION-PRESSURE IN AN EXPANDING ENVELOPE.

B. P. Gerasimovič.

It is well known that internal relative motions in nebular strata make them more transparent to their own radiation, thus tending to reduce the radiation-pressure in the envelope. This decrease in momentum can be of importance in the mechanics of planetary nebulæ, steady outflow in Be stars, Novæ outbursts, etc. If sufficiently large, these effects of relative motions can change the outflow altogether, reducing the braking action of L_α in non-static chromospheres of the P Cygni kind and diminishing the otherwise unacceptably large accelerations in planetaries.

There are some qualitative physical arguments which allow us to think that this decrease in momentum in L_α radiation caused by relative motion cannot be large in the general case. In fact, in nebular conditions owing to the high absorption coefficient for L_α radiation, the optical thickness in these frequencies should be very large even if the optical thickness in continuous Lyman frequencies is reasonably moderate. This intrinsic opacity for L_α means that the atoms are subject to the L_α radiation emitted by the neighbouring atoms, which have quite negligible relative mass-velocities, even if the change in systematic velocity through the nebula as a whole is not small. If this argument is correct, we may calculate the radiation-pressure for L_α in moving envelopes, using the same method as in a static case, thus greatly simplifying the problem. Such a reasoning has been adopted by the present writer in his study * of the steady outflow in Be stars ; it was, however, realized that more detailed calculations of the effect in question are required in order to verify the above considerations.

These calculations seem to be of interest also for another reason. The effect of relative motion on radiation-pressure in some special cases has been considered by Ambarzumian,† Woltjer ‡ and Zanstra.§ It can be

* B. Gerasimovič, *M.N.*, **94**, 737, 1934.
† V. Ambarzumian, *Bull. Poulkovo*, **13**, 3, 1933.
‡ J. Woltjer, jun., *B.A.N.*, **7**, 7, 1934.
§ H. Zanstra, *M.N.*, **95**, 84, 1934.

shown, however, that, if only *the order* of the expected effect is desired, the problem admits a simple solution, which, being independent of the solution of the equations of transfer, seems to be sufficiently general.

This note deals with the calculation of the upper limit of the effect in question, using a simple kinematical method of reasoning. It is shown that except for some special cases the reduction in radiation-pressure in L_α frequencies, caused by the expansion of nebular envelope, is generally small (*e.g.* smaller than 1 per cent. in a case considered).

In Novæ envelopes, where relative velocities of 10^3 km. and more are to be expected, this reduction in radiation-pressure can be appreciable.

The problem consists in calculating the momentum gained by the element of volume at A from the L_α radiation emitted by the corresponding element at B, supposing that A moves with respect to B with a constant velocity V in the direction BA. Let the non-accented symbols refer to B, the same symbols accented, to the observer at A. Considering first the three-dimensional case, we have the following three transformation formulæ for the frequency ν, angles θ and ω and the intensity of radiation I:—

$$\nu' = \nu\left(1 - \frac{V}{c}\cos\theta\right)$$

$$\cos\theta' = \cos\theta - \frac{V}{c} + \frac{V}{c}\cos^2\theta$$

$$\frac{I'(\nu'\theta')}{I(\nu,\theta)}\frac{d\nu' d\omega'}{d\nu d\omega} = 1 - 2\frac{V}{c}\cos\theta.$$

The first is Doppler's formula, the second the aberration formula, and the third the transformation formula for the intensity,* these transformations being correct to the first order of V/c. Now

$$\frac{d\omega}{d\omega'} = \frac{d\cos\theta}{d\cos\theta'} = 1 - 2\frac{V}{c}\cos\theta,$$

and therefore

$$I'(\nu', \theta') = \left(1 - 3\frac{V}{c}\cos\theta\right)I(\nu, \theta).$$

If the emission is isotropic,

$$I(\nu, \theta) = I(\nu) = I\left(\nu' + \frac{V}{c}\nu'\cos\theta'\right)$$

or

$$I'(\nu', \theta') = \left(1 - 3\frac{V}{c}\cos\theta'\right)I\left(\nu' + \frac{V}{c}\nu'\cos\theta'\right).$$

In the one-dimensional case, which will be considered here,

$$I'(\nu') = \left(1 - \frac{V}{c}\right)I\left(\nu' + \frac{V}{c}\nu'\right).$$

---

* Compare, *e.g.*, A. Einstein, *Phys. Zeitschrift*, **18**, 125, 1917, or E. A. Milne, *M.N.*, **86**, 582, 1926.

If there are no absorbing atoms between B and A, the required momentum gained by A from B should be proportional to

$$\int \kappa(\nu') I(\nu') d\nu',$$

where $\kappa(\nu')$ is the absorption coefficient for ν'-radiation, which in nebular conditions can be taken as

$$\kappa(\nu') = \frac{\text{const}}{\nu'} \cdot e^{-\frac{c^2}{v_t^2}\left(\frac{\nu'-\nu_0}{\nu_0}\right)^2},$$

v_t denoting thermal velocity. On the other hand, the intensity of ν-radiation at B is

$$I(\nu) = P(B) e^{-\frac{c^2}{v_t^2}\left(\frac{\nu-\nu_0}{\nu_0}\right)^2},$$

with a proper normalization factor. Thus for ν' between ν' and $\nu'+d\nu'$ the momentum at A is proportional to

$$P(B) e^{-\frac{c^2}{v_t^2}\left(\frac{\nu'+\nu'\frac{V}{c}-\nu_0}{\nu_0}\right)^2} \cdot e^{-\frac{c^2}{v_t^2}\left(\frac{\nu'-\nu_0}{\nu_0}\right)^2},$$

putting $1-\frac{V}{c} \sim 1$ under nebular conditions.

If there are absorbing atoms between A and B (continuous nebula), this expression should be multiplied by a factor depending on the corresponding absorption of $\nu' \ldots \nu'+d\nu'$-radiation, which makes the transmitted intensity

$$I'(\nu') e^{-(s_{\nu'}-\sigma_{1\nu'})} d\nu',$$

where $\sigma_{\nu'}$ is the optical thickness for the radiation, which is observed at A as ν'-radiation ; $s_{\nu'}$ and $\sigma_{1\nu'}$ are the values of $\sigma_{\nu'}$ at A and B (see diagram).

There are two different conceptions of optical thickness in a dynamical case. First, the above $\sigma_{\nu'}$, *i.e.* the optical thickness "following the radiation" ; second, $\tau_{\nu'}$, that for the radiation which is observed as ν' *everywhere* in the envelope. In a stationary envelope, of course, $\sigma_{\nu'} = \tau_{\nu'}$. Now at C according to definition

$$\frac{d\sigma_{\nu'}}{d\tau_{\nu'}} = \frac{\kappa(\nu)}{\kappa(\nu')},$$

where $\nu = \nu'\left(1+\frac{v}{c}\right)$ and v is the velocity of A with respect to C. Thus

$$\frac{d\sigma_{\nu'}}{d\tau_{\nu'}} = \left(1-\frac{v}{c}\right) e^{-\frac{c^2}{v_t^2}\left(\frac{\nu'+\nu'\frac{v}{c}-\nu_0}{\nu_0}\right)^2} \Big/ e^{-\frac{c^2}{v_t^2}\left(\frac{\nu'-\nu_0}{\nu_0}\right)^2},$$

where $1-\frac{v}{c} \sim 1$.

To calculate the *order* of the change in radiation-pressure produced by the expansion, let us consider only the centre of the line $\nu' = \nu_0$. This approximation is fully justifiable, since owing to two exponential factors

the resulting momentum decreases very rapidly for $\nu' \neq \nu_0$. In the case under consideration,

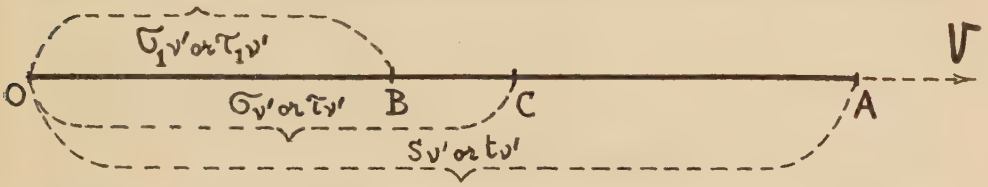
$$d\sigma = e^{-w^2} d\tau,$$

where $w = \frac{v}{v_t}$, the ν'-suffices being omitted.

Thus

$$s - \sigma = \int_{\tau}^{t} e^{-w^2} d\tau$$

(t being the value of τ at A). It is seen that always $s - \sigma \leqslant t - \tau$, which is due to the increase in transparency caused by the motion. In such a case



the momentum in A from B (in BA direction) in the range $(\nu_0, \nu_0 + d\nu_0)$ of frequency is proportional to

$$P(B) e^{-w^2 - \int_{\tau_1}^{t} e^{-w^2} d\tau} d\nu_0 \sim f(\tau_1) e^{-w^2 - \int_{\tau_1}^{t} e^{-w^2} d\tau} d\nu_0,$$

where $f(\tau_1)$ is the *Ergiebigkeit* at B, and τ_1 is the corresponding value of τ. The ratio of this momentum to the corresponding momentum for a stationary envelope $(f(\tau_1) e^{-(t-\tau_1)})$ is

$$q = \exp\left[-\left(w^2 + \int_{\tau_1}^{t} e^{-w^2} d\tau - (t - \tau_1) \right) \right],$$

independently of the change of intensity, etc., inside the envelope. If the *Ergiebigkeit* is known, the total momentum gained by A in both stationary and dynamical cases can be calculated upon proper integration. Avoiding unnecessary hypotheses and aiming at the calculation of the *order* of magnitude, we can see the difference between both cases by simply calculating the ratio of partial momenta or q for various values of $(t - \tau)$ from 0 to say 1, and for various assumptions about the function

$$w = \phi(t - \tau_1).$$

Take the simplest supposition,

$$w^2 = a(t - \tau_1),$$

when

$$q = \exp\left[-\left(w^2 + (t - \tau_1) \frac{(1 - e^{-w^2} - w^2)}{w^2} \right) \right].$$

Consider now an extreme case; the velocity changing from 0 to 200 km./sec.$^{-1}$ as the corresponding optical thickness in continuous Lyman radiation changes from 0 to 2. Since the ratio of absorption coefficients

for L_α and for continuous Lyman radiation is $6{\cdot}7 \cdot 10^3$, the optical thickness in L_α should change from 0 to $13{\cdot}4 \cdot 10^3$. For a B-type hydrogen envelope we can take $v_t = 20$ km./sec.$^{-1}$, so that for two extreme parts of the envelope $w = 10$. These assumptions give

$$a = 10^2/13{\cdot}4 \cdot 10^3 = 7{\cdot}5 \cdot 10^{-3}.$$

To estimate the effect of radiation-pressure in L_α, take, for example, $t - \tau_1 = 1$ (except for a very special distribution of $f(\tau)$ in the envelope the values of $t - \tau_1$ greater than say 2 are for obvious reasons of no importance in the integrated momentum). Then for two points separated by optical distance in L_α frequencies equal to 1, the corresponding $w^2 = 7{\cdot}5 \cdot 10^{-3}$ and

$$q = \exp\left[-3{\cdot}4 \times 10^{-3}\right],$$

corresponding to a decrease in partial momentum of less than 1 per cent.* It seems unnecessary to calculate q for $t - \tau_1 \gtrless 1$, since in all cases the *order* of the effect in question remains practically the same.

Consider another case—likely to occur in Novæ—the velocity changing from 0 to 10^3 km. and v_t as before, and the optical thickness in Lyman continuous radiation 2. Here $a = 0{\cdot}19$, and for $(t - \tau_1) = 1$,

$$q = \exp\left[-0{\cdot}10\right],$$

i.e. the reduction in partial momentum is about 10 per cent. The order of the calculated effect will not be much changed for other laws of w, e.g. $w = a(t - \tau_1)$, instead of parabolic dependence.

All these considerations show that in actual nebulæ or nebular envelopes the radiation-pressure due to L_α cannot be reduced very considerably by the expansion, so that the resulting momentum remains practically the same as in a static case.† This result is at variance with Zanstra's recent conclusions (*op. cit.*) reached after solving the equations of transfer for an expanding envelope. This is probably due to a large value (10 km.) which Zanstra adopts for v_L, *i.e.* for the range in velocity within which an appreciable fraction of the luminosity is concentrated, and which causes the observed Doppler broadening of the line due to mass motion, if the thermal width is small. Though this value of v_L agrees with the observed widths of Balmer lines, it can hardly represent v_L for L_α. In fact, owing to the lack of transparency of the nebula in L_α frequencies, the real range in v_L within which an appreciable part of the luminosity in L_α is concentrated should be very small—certainly much smaller than the thermal velocity.‡

Though the absorption coefficients for Balmer lines are not much

* For small a the value $t - \tau_1 = 1$ corresponds to the minimum of q.

† Without considering the limiting case $v_t \to 0$ when $w \to \infty$ and $q \to 0$. This case has no physical significance.

‡ The present method consists in comparing the radiation-pressure in L_α for expanding and stationary envelopes with the same distribution of $f(\tau)$. Of course the reduction in absorption caused by the expansion affects re-emission and therefore the function $f(\tau)$, these effects being, however, of a secondary importance as soon as a steady state is reached.

smaller than that for L_α, the nebula will be more or less transparent in Balmer frequencies, owing to a very small concentration of atoms in the second state. In this way the effect of expansion on radiation-pressure in Balmer frequencies should be relatively larger than that for L_α. It goes without saying that the corresponding macroscopic effects are of no mechanical importance.

Poulkovo :
1935 *December.*

SOLUTIONS OF THE EQUATIONS OF VARIATIONS.

Ernest W. Brown.

1. Let the equations of motion of a dynamical problem be

$$\ddot{X}+\frac{\partial F}{\partial X}=\alpha X_\alpha, \qquad \ddot{Y}+\frac{\partial F}{\partial Y}=\alpha Y_\alpha, \qquad \ddot{Z}+\frac{\partial F}{\partial Z}=\alpha Z_\alpha, \tag{1.1}$$

where α is a constant not present explicitly in F, X_α, Y_α, Z_α.

When $\alpha=0$, let the general solution containing the six arbitrary constants c_i $(i=0, 1, \ldots 5)$ be denoted by $X=X_0$, $Y=Y_0$, $Z=Z_0$. Put

$$\frac{dX}{d\alpha}=x, \qquad \frac{dX_0}{dc_i}=x_i,$$

with similar definitions of y, z, y_i, z_i.

If (1.1) be differentiated with respect to α, we obtain

$$\left.\begin{aligned} \ddot{x}+F_{11}x+F_{12}y+F_{13}z&=X_\alpha,\\ \ddot{y}+F_{21}x+F_{22}y+F_{23}z&=Y_\alpha,\\ \ddot{z}+F_{31}x+F_{32}y+F_{33}z&=Z_\alpha,\end{aligned}\right\} \tag{1.2}$$

where it is implicitly assumed that the right-hand members of (1.1) are independent of X, Y, Z. The F_{ij} are the second derivatives of F with respect to X, Y, Z.

If we put $\alpha=0$ and differentiate (1.1) with respect to c_i, we obtain

$$\left.\begin{aligned} \ddot{x}_i+F_{11}x_i+F_{12}y_i+F_{13}z_i&=0,\\ \ddot{y}_i+F_{21}x_i+F_{22}y_i+F_{23}z_i&=0,\\ \ddot{z}_i+F_{31}x_i+F_{32}y_i+F_{33}z_i&=0.\end{aligned}\right\} \tag{1.3}$$

The F_{ij} in (1.3) are the same functions of X, Y, Z as those in (1.2) but do not have the same values, since $\alpha=0$ in (1.3). If, however, we neglect squares of α in (1.1), the F_{ij} in (1.2) will be independent of α and therefore have the same values as in (1.3). This assumption will be adopted below. With it the equations (1.2) are valid when X_α, Y_α, Z_α are functions of X, Y, Z, for they now become functions of X_0, Y_0, Z_0.

2. The problem considered below is the development of certain types of solutions of (1.3). With the usual method, that of the variation of the

arbitraries, X_a, Y_a, Z_a are the derivatives with respect of X, Y, Z of a certain function R, and, with a proper choice of the c_i, the solution is reduced to that of the equations

$$\frac{dc_i}{dt}=\frac{\partial R}{\partial c_{i+1}}, \qquad \frac{dc_{i+1}}{dt}=-\frac{\partial R}{\partial c_i}, \qquad i=0,\ 2,\ 4.$$

A slight extension of this, in which we replace

$$\frac{\partial R}{\partial c_i} \quad \text{by} \quad \frac{dK}{dt}\equiv X_a x_i+Y_a y_i+Z_a z_i, \qquad i=0,\ 1\ \ldots\ 5,$$

permits the method to be applied to (1.2).

Since we are, in effect, neglecting the variable parts of the c_i in R, we have

$$\sum_i\left(\frac{\partial X}{\partial c_i}\delta c_i+\frac{\partial X}{\partial c_{i+1}}\delta c_{i+1}\right)=\sum_i(x_iK_{i+1}-x_{i+1}K_i), \qquad i=0,\ 2,\ 4,$$

with similar expressions for y, z.

3. The practical application of these formulæ to problems in celestial mechanics demands the expression of X_0, Y_0, Z_0 as literal functions of the c_0. Three of the constants are usually taken to be angles, while the other three are functions of n (mean motion), e (eccentricity), i (inclination). As long as the solution X_0, Y_0, Z_0 is purely elliptic the six derivatives are easily found and are well known. But there are other cases, notably some of those in which a fourth body disturbs a three-body system, where this solution cannot be assumed to be elliptic without serious error. In order to obtain sufficient accuracy in finding X_0, Y_0, Z_0 it is usually necessary to substitute the numerical value of n or a at the outset. If this be done a new calculation must be undertaken to obtain the derivative with respect to one of these constants. It is usually not difficult to find the derivatives with respect to the remaining constants. The main object of this paper is the discovery of a solution which does not require derivatives with respect to n or a, and which is in a form convenient for calculation.

In a paper published * in 1900 I gave a partial solution of the problem which served the purpose for which it was developed. This solution, however, was limited to the first power of the eccentricity, and the inclination was neglected in X_0, Y_0, Z_0. The problem has arisen in a general form with the proposal to verify the whole of the solar action on the Moon's motion by Professor W. J. Eckert and myself, with the aid of computing machines of the Hollerith type. Here αX_a, αY_a, αZ_a will be residuals after the previously calculated values of the co-ordinates have been substituted in the equations of motion. Although the method finally adopted for the verification and extension to a higher degree of accuracy will probably not require the use of the formulæ obtained below, they are useful in approximating to the motion of any satellite or, as already pointed out, to the calculation of the action of a fourth body on a three-body system.

* *Camb. Phil. Trans.*, **18**, 94–106.

4. As the chief application of the formulæ to be obtained will be to satellite motion, let us so choose the co-ordinates and arbitrary constants that X_0 consists of sums of cosines and Y_0, Z_0 of sums of sines of multiples of angles, the constant parts of which are the arbitraries c_1, c_3, c_5, and of other angles which are independent of the arbitraries. Let the arbitraries c_0, c_2, c_4 be parameters. Denote the constant motions of the angles whose constant parts are c_1, c_3, c_5 by n_1, n_3, n_5.

To abbreviate the exposition, let us adopt the notations,

$$(p, p_i) \equiv \dot{x}x_i + \dot{y}y_i + \dot{z}z_i - x\dot{x}_i - y\dot{y}_i - z\dot{z}_i, \tag{4.1}$$

$$K_i \equiv \int (X_\alpha x_i + Y_\alpha y_i + Z_\alpha z_i)dt. \tag{4.2}$$

It is assumed that x, y, z vanish with X_α, Y_α, Z_α, so that it is not necessary to add any constant to the integrals.

Multiply the three equations (1.2) by x_i, y_i, z_i, and subtract from the sum of the products the sum of the products of (1.3) by x, y, z. The F_{ij} disappear and the result is integrable, giving

$$(p, p_i) = K_i. \tag{4.3}$$

Follow a similar process with the factors x_i, y_i, z_i and x_j, y_j, z_j and the equations (1.3) with suffixes i, j. We obtain

$$(p_j, p_i) = \text{const.} = C_{ji}. \tag{4.4}$$

The definitions of the co-ordinates and constants show that (p_j, p_i) will consist of sines when j, i are both even or both odd. Hence

$$C_{ji} = 0, \qquad j, i = 0, 2, 4, \quad \text{or} \quad j, i = 1, 3, 5. \tag{4.5}$$

For any three suffixes i, j, k, the identity

$$\frac{dC_{ij}}{dc_k} + \frac{dC_{jk}}{dc_i} + \frac{dC_{ki}}{dc_j} = 0$$

is easily proved by direct differentiation. Let i, j be even and k odd. Then since $C_{ij} = 0$, and since $C_{jk} = -C_{kj}$, we have

$$\frac{dC_{kj}}{dc_i} = \frac{dC_{ki}}{dc_j}.$$

The C_{ki} are therefore derivatives with respect to c_i of a constant B_k. But the constants C_{ji} cannot contain the angles c_1, c_3, c_5. Hence

$$C_{ji} = \frac{dB_j}{dc_i}, \qquad j = 1, 3, 5; \qquad i = 0, 2, 4.$$

It is easy to see that B_1, B_3, B_5 are the canonical constants complementary to c_1, c_3, c_5. For if we put $B_1 = c_0$, $B_3 = c_2$, $B_5 = c_4$, all the C_{ji} vanish except C_{10}, C_{32}, C_{54}, which are unity.

5. As stated above, the object of this paper is to obtain a solution of (1.2) without the use of one of the derivatives which will be taken to be that with respect to c_0. Since (1.2) are completely replaced by (4.3), the

problem is the discovery of the general solution of (4.3) without the use of x_0, y_0, z_0.

Two forms of the solution will be given. If the real co-ordinates x, y, z be used, it is found that x_1, y_1, z_1 appear as denominators of certain fractions present in the solution. As each of these can vanish, the result is useless for calculation. As far as x, y are concerned, the difficulty is avoided by the use of the complex co-ordinates,

$$u = x + y\sqrt{-1}, \qquad s = x - y\sqrt{-1}.$$

The similar difficulty in the expression for z is avoided in the second solution.

6. With u, s replacing x, y, equations (2.1), (2.4) become

$$\dot{u}s_i + \dot{s}u_i - u\dot{s}_i - s\dot{u}_i + 2\dot{z}z_i - 2z\dot{z}_i = 2K_i, \tag{6.1}$$

$$\dot{u}_j s_i + \dot{s}_j u_i - u_j \dot{s}_i - s_j \dot{u}_i + 2\dot{z}_j z_i - 2z_j \dot{z}_i = 2C_{ji}. \tag{6.2}$$

The computation of the K_i, C_{ji} can be made with the real variables.

With $i = 1, 2, 3, 4, 5$ there are five of the equations (6.1). From them eliminate $\dot{s}, s, \dot{z}, z$. We obtain

$$\dot{u}\Delta - u\dot{\Delta} = 2\Sigma_i K_i \Delta_i, \tag{6.3}$$

where

$$\Delta = |\, s_i, u_i, \dot{u}_i, z_i, \dot{z}_i \,|, \tag{6.4}$$

and $(-1)^{i+1}\Delta_i$ is the first minor of s_i in Δ. It is easily seen that

$$\dot{\Delta} = |\, \dot{s}_i, u_i, \dot{u}_i, z_i, \dot{z}_i \,|.$$

For if we differentiate Δ by adding the determinants obtained by differentiating the successive columns, the second and fourth vanish because they have two columns with the same elements, and the third and fifth vanish because $\ddot{u}_i, \ddot{z}_i$ are linear functions of u_i, s_i, z_i with coefficients which are the same for all values of i.

Since K_i is replaced by C_{ji} when u, s, z are replaced by u_j, s_j, z_j, (6.3) gives

$$\dot{u}_j\Delta - u_j\dot{\Delta} = 2\Sigma_i C_{ji}\Delta_i. \tag{6.5}$$

Since $C_{ji} = 0$ when j, i are both even or both odd, the equations (6.5) are separable into two sets. When $j = 1, 3, 5$ only Δ_2, Δ_4 of the five Δ_i are present, and if we eliminate them from the equations we obtain

$$|\, \dot{u}_j\Delta - u_j\dot{\Delta}, C_{j2}, C_{j4} \,| = 0. \qquad j = 1, 3, 5.$$

Hence, integrating,

$$\Delta = |\, u_j, C_{j2}, C_{j4} \,|, \qquad j = 1, 3, 5. \tag{6.6}$$

The arbitrary constant has been put equal to unity since it divides out when the values of Δ, Δ_i are substituted in (6.3). When Δ has been found, the values of Δ_2, Δ_4 are given by

$$\left.\begin{aligned} 2C_{32}\Delta_2 + 2C_{34}\Delta_4 &= \dot{u}_3\Delta - u\dot{\Delta}_3, \\ 2C_{52}\Delta_2 + 2C_{54}\Delta_4 &= \dot{u}_5\Delta - u\dot{\Delta}_5. \end{aligned}\right\} \tag{6.7}$$

With $j=2, 4$ in (6.5), we obtain

$$\left.\begin{aligned} 2C_{21}\Delta_1+2C_{23}\Delta_3+2C_{25}\Delta_5&=\dot{u}_2\Delta-u_2\dot{\Delta},\\ 2C_{41}\Delta_1+2C_{43}\Delta_3+2C_{45}\Delta_5&=\dot{u}_4\Delta-u_4\dot{\Delta}.\end{aligned}\right\} \tag{6.8}$$

The additional equation necessary for finding Δ_1, Δ_3, Δ_5 is given by the identity,

$$\Sigma_i u_i\Delta_i=0, \qquad i=1, 2, 3, 4, 5. \tag{6.9}$$

7. *Changes of Variables.*—If we put C_1 for the coefficient of u_1 in (6.6), we have

$$C_1=C_{32}C_{54}-C_{52}C_{34}. \tag{7.1}$$

I shall not stop to prove that C_1 is never zero. It may be pointed out, however, that it is unity when canonical constants are used.

Change from u_1 to u_1' defined by

$$u_1'\equiv u_1-\frac{u_3}{C_1}(C_{12}C_{54}-C_{52}C_{14})+\frac{u_5}{C_1}(C_{12}C_{34}-C_{32}C_{14}), \tag{7.2}$$

with similar definitions of s_1', z_1'. From these we form C_{ij}' and find that

$$C_{12}'=C_{14}'=0.$$

Next, change u_3, u_5 to u_3', u_5' defined by

$$u_3=C_{32}u_3'+C_{34}u_5', \qquad u_5=C_{52}u_3'+C_{54}u_5', \tag{7.3}$$

with similar changes for s, z. Substituting in (6.2) we obtain the relations

$$\begin{aligned} C_{32}&=C_{32}C_{32}'+C_{34}C_{52}', & C_{34}&=C_{32}C_{34}'+C_{34}C_{54}',\\ C_{52}&=C_{52}C_{32}'+C_{54}C_{52}', & C_{54}&=C_{52}C_{34}'+C_{54}C_{52}',\end{aligned}$$

which give

$$C_{34}'=C_{52}'=0, \qquad C_{32}'=C_{54}'=1.$$

Thus all the C_{ij}' are zero except C_{32}', C_{54}', which have the value unity.

A similar result is obtained if instead of the above we change to u_2', u_4', defined by

$$u_2=C_{23}u_2'+C_{25}u_4', \qquad u_4=C_{43}u_2'+C_{45}u_4',$$

which give $C_{34}'=C_{52}'=0$, $C_{32}'=C_{54}'=-1$. As u_1 has to be changed in any case it is better to change u_3, u_5 rather than u_2, u_4 for the sake of uniformity.

Since the new variables are linear functions of the old with constant coefficients they are equally solutions of the equations of variations. I shall adopt them and, for brevity, drop the accents.

8. With the new variables,

$$\begin{aligned} \Delta&=u_1, & 2\Delta_2&=\dot{u}_3u_1-u_3\dot{u}_1, & 2\Delta_4&=\dot{u}_5u_1-u_5\dot{u}_1,\\ & & 2\Delta_3&=-(\dot{u}_2u_1-u_2\dot{u}_1), & 2\Delta_5&=-(\dot{u}_4u_1-u_4\dot{u}_1).\end{aligned}$$

The identity $\Sigma u_i\Delta_i=0$ gives

$$2\Delta_1=\dot{u}_2u_3-u_2\dot{u}_3+\dot{u}_4u_5-u_4\dot{u}_5.$$

Hence, substituting in (6.3), we obtain

$$\frac{d}{dt}\left(\frac{u}{u_1}\right)=\frac{K_1}{u_1^{\,2}}(\dot u_2u_3-u_2\dot u_3+\dot u_4u_5-u_4\dot u_5)$$
$$+K_2\frac{d}{dt}\left(\frac{u_3}{u_1}\right)-K_3\frac{d}{dt}\left(\frac{u_2}{u_1}\right)+K_4\frac{d}{dt}\left(\frac{u_5}{u_1}\right)-K_5\frac{d}{dt}\left(\frac{u_4}{u_1}\right),$$

with similar expressions for the derivatives of s/s_1, z/z_1.

This is the first solution. Its symmetry shows how it might be extended to any number of variables. As mentioned earlier, the formula for z cannot be used for calculation on account of the vanishing of z_1 for certain values of t.

9. *Second Solution.*—The new variables, omitting the accents, will be adopted. If instead of eliminating from (6.1) all unknowns but $\dot u$, u, we eliminate all but u, s and obtain

$$-s\Delta_u-u\Delta_s=2\Sigma K_i\Delta_i,\qquad i=1,\ 2,\ 3,\ 4,\ 5, \tag{9.1}$$

where

$$\Delta_u=|\,\dot u_i,\ u_i,\ s_i,\ \dot z_i,\ z_i\,|\,,\qquad \Delta_s=|\,\dot s_i,\ u_i,\ s_i,\ \dot z_i,\ z_i\,|\,,$$

$(-1)^{i+1}\Delta_i=$first minor of $\dot u_i$ in Δ_u.

Let the first minors of s_i in Δ_u be denoted by $(-1)^{i+1}d_i$. Then, by arguments similar to those in § 6,

$$\Delta_u=\Sigma s_id_i,\qquad \dot\Delta_u=\Sigma\dot s_id_i,\qquad 0=\Sigma\dot u_id_i=\Sigma u_id_i=\Sigma\dot z_id_i=\Sigma z_id_i.$$

With the aid of the relations (6.2) these give

$$\dot u_j\Delta_u-\dot u_j\dot\Delta_u=2\Sigma C_{ji}d_i.$$

Since all the C_{1j} are zero, the right-hand member of this vanishes when $i=1$. Hence, on integration, $\Delta_u=ku_1$. Thence by putting $u=u_1$, $s=s_1$, $K_i=C_{1i}$ in (9.1) we obtain $\Delta_s=-ks_1$.

With these we have, by arguments similar to those previously used,

$$ku_1=\Sigma\dot u_i\Delta_i,\qquad -ks_1=\Sigma\dot s_i\Delta_i,\qquad 0=\Sigma u_i\Delta_i=\Sigma s_i\Delta_i=\Sigma\dot z_i\Delta_i=\Sigma z_i\Delta_i.$$

Whence, proceeding as before,

$$k(s_ju_1-u_js_1)=2\Sigma C_{ij}\Delta_i$$

with $j=2,\ 3,\ 4,\ 5$, this gives

$$2\Delta_2=k(u_3s_1-s_3u_1),\qquad 2\Delta_4=k(u_5s_1-s_5u_1),$$
$$2\Delta_3=-k(u_2s_1-s_2u_1),\qquad 2\Delta_5=-k(u_4s_1-s_4u_1),$$

and the identity $\Sigma u_i\Delta_i=0$ then gives

$$2\Delta_1=k(u_2s_3-s_2u_3+u_4s_5-s_4u_5).$$

Substituting these values of the Δ_i, Δ_u, Δ_5 in (9.1), we obtain

$$\frac{u}{u_1}-\frac{s}{s_1}=\frac{K_1}{u_1s_1}(u_2s_3-s_2u_3+u_4s_5-s_4u_5)$$
$$+K_2\left(\frac{u_3}{u_1}-\frac{s_3}{s_1}\right)-K_3\left(\frac{u_2}{u_1}-\frac{s_2}{s_1}\right)+K_4\left(\frac{u_5}{u_1}-\frac{s_5}{s_1}\right)-K_5\left(\frac{u_4}{u_1}-\frac{s_4}{s_1}\right). \tag{9.2}$$

10. Equation (6.1) with $i=1$ can be written

$$\frac{d}{dt}\left(\frac{u}{u_1}+\frac{s}{s_1}\right)+\left(\frac{u}{u_1}-\frac{s}{s_1}\right)\left(\frac{\dot{u}_1}{u_1}-\frac{\dot{s}_1}{s_1}\right)+\frac{2\dot{z}z_1-2z\dot{z}_1}{u_1s_1}=\frac{K_1}{u_1s_1}. \tag{10.1}$$

By eliminating from (6.1) all the variables but $\dot{z}$, z and following a procedure like that used earlier, we obtain

$$\begin{aligned}\dot{z}z_1-z\dot{z}_1=&K_1(\dot{z}_2z_3-z_2\dot{z}_3+\dot{z}_4z_5-z_4\dot{z}_5)\\&+K_2(\dot{z}_3z_1-z_3\dot{z}_1)-K_3(\dot{z}_2z_1-z_2\dot{z}_1)\\&+K_4(\dot{z}_5z_1-z_5\dot{z}_1)-K_5(\dot{z}_4z_1-z_4\dot{z}_1).\end{aligned} \tag{10.2}$$

When $u/u_1-s/s_1$ has been obtained from (9.2), we obtain $u/u_1+s/s_1$ from (10.1) with (10.2).

It can be immediately seen that the equations (9.2), (10.1) give the real co-ordinates. For if, before the change of variables, we choose

$$c_1=\epsilon, \qquad c_3=\epsilon-\varpi, \qquad c_5=\epsilon-\theta,$$

where ϵ, ϖ, θ are the constants of the mean longitude, of the perigee and of the node, and remember that the principal term in u is $a\exp\iota(nt+\epsilon)$, we see that the principal terms in u_1, s_1 are

$$a\iota\exp\iota(nt+\epsilon), \qquad -a\iota\exp-\iota(nt+\epsilon).$$

Thus the principal term in $u/u_1-s/s_1$ is $-2x\iota$ and that in $u/u_1+s/s_1$ is $2y$, where x, y are co-ordinates referred to axes following the mean angular position of the body. The value of x is first determined from (9.2) and requires only single integrations. When the result is substituted in (10.1) it is multiplied by the small factor $\dot{u}_1/u_1-\dot{s}_1/s_1$.

11. To determine z, we eliminate from (6.1) all the unknowns but z, u and follow a procedure similar to that of § 9. The result may be written

$$\begin{aligned}z-z_1\frac{u}{u_1}=&\frac{K_1}{u_1}(z_2u_3-z_3u_2+z_4u_5-z_5u_4)\\&+K_2\left(z_3-z_1\frac{u_3}{u_1}\right)-K_3\left(z_2-z_1\frac{u_2}{u_1}\right)\\&+K_4\left(z_5-z_1\frac{u_5}{u_1}\right)-K_5\left(z_4-z_1\frac{u_4}{u_1}\right).\end{aligned} \tag{11.1}$$

An equation with s, s_i replacing u, u_i can also be obtained. The most convenient form for calculation is the equation which is the sum of these. It is to be noticed that z_1 then has the disturbing force as a factor so that inaccuracies in $u/u_1+s/s_1$ will have a minor effect on z.

12. In order that these solutions may contain only harmonic terms, it is necessary to take account of the fact that c_2, c_4 are present in the motions of the angles n_3t+c_3, n_5t+c_5, as well as in the coefficients of the harmonic terms, so that when derivatives with respect to c_2, c_4 are formed, terms factored by t will appear.

The terms so arising in any co-ordinate q are

in q_2: $$t\frac{dn_3}{dc_2}q_3+t\frac{dn_5}{dc_2}q_5,$$

in q_4: $$t\frac{dn_3}{dc_4}q_3+t\frac{dn_5}{dc_4}q_5.$$

Let us insert these additional terms in (9.2). The existence of the relation $dn_5/dc_2=dn_3/dc_4$ causes the additional terms to disappear from the coefficient of K_1. Inserting them in K_2 and integrating by parts, we find that the addition to K_2 is

$$t\frac{dn_3}{dc_2}K_3+t\frac{dn_5}{dc_2}K_5-\frac{dn_3}{dc_2}\int K_3dt-\frac{dn_5}{dc_2}\int K_5dt.$$

If these, together with the corresponding terms in K_4, u_2, s_2, u_4, s_4, be inserted in the remaining portion of (9.1), the terms factored by t will disappear. A similar result holds for (10.1), (11.1).

Hence we may form derivatives with respect to c_2, c_4 from the coefficients only of the periodic terms if we add

$$-\frac{dn_3}{dc_2}\int K_3dt-\frac{dn_5}{dc_2}\int K_5dt \quad \text{to } K_2,$$

$$-\frac{dn_3}{dc_4}\int K_3dt-\frac{dn_5}{dc_4}\int K_5dt \quad \text{to } K_4.$$

An examination of the processes which lead up to (9.2), (10.1), (11.1) shows that none of them is invalidated by the presence of these terms.

13. The functions $\dot{K}_2$, $\dot{K}_4$ will in general contain constant terms k_2, k_4, which, on integration, will produce terms factored by t in the co-ordinates. These can be avoided by supposing that in the solution X_0, Y_0, Z_0 the motions n_3, n_5 receive additional portions δn_3, δn_5.

The additional portion of the left member of (9.2) will be

$$\delta n_3 t\left(\frac{u_3}{u_1}-\frac{s_3}{s_1}\right)+\delta n_5 t\left(\frac{u_3}{u_1}-\frac{s_5}{s_1}\right).$$

That on the right is

$$k_2 t\left(\frac{u_3}{u_1}-\frac{s_3}{s_1}\right)+k_4 t\left(\frac{u_5}{u_1}-\frac{s_5}{s_1}\right).$$

These disappear if

$$\delta n_3=k_2, \qquad \delta n_5=k_4.$$

The same values evidently satisfy (10.2), (11.1).

Substitute these additional portions in (10.1). It is immediately seen that the coefficient of $t\delta n_3$ is C_{31}/u_1s_1 and that of $t\delta n_5$ is C_{51}/u_1s_1. These therefore vanish. The remaining terms to be added to the left member of (10.1) are

$$[\delta n_3(u_3s_1+s_3u_1+2z_3z_1)+\delta n_5(u_5s_1+s_5u_1+2z_5z_1)]\div u_1s_1.$$

This completes the solution of the problem.

14. Considerable simplifications of the formulæ can be introduced if we can treat the equations for x, y or u, s separately from that for z. If we confine x, y to the terms independent of the inclination and z to those dependent on its first power there is no difficulty, since the neglected terms in x, y depend on the square of the inclination and those in z on its cube.

Equations (6.1), (6.2) become

$$\dot{u}s_i + \dot{s}\dot{u}_i - u\dot{s}_i - s\dot{u}_i = 2K_i = 2\int\left(X_\alpha\frac{dX_0}{dc_i} + Y_\alpha\frac{dY_0}{dc_i}\right)dt,$$

$$\dot{u}_j s_i + \dot{s}_j u_i - u_j\dot{s}_i - s_j\dot{u}_i = 2C_{ji}.$$

Here $i, j = 1, 2, 3$, with $C_{13} = 0$.

The transformed variables are

$$u_1' = u_1 - u_3\frac{C_{12}}{C_{32}}, \qquad u_3 = C_{32}u_3',$$

giving

$$C_{12}' = 0, \qquad C_{32}' = 1.$$

Inserting these in (9.1), (10.2), and dropping the accents, we obtain

$$\frac{u}{u_1} - \frac{s}{s_1} = \frac{K_1}{u_1 s_1}(u_2 s_3 - s_2 u_3) + K_2\left(\frac{u_3}{u_1} - \frac{s_3}{s_1}\right) - K_3\left(\frac{u_2}{u_1} - \frac{s_2}{s_1}\right), \tag{14.1}$$

$$\frac{d}{dt}\left(\frac{u}{u_1} + \frac{s}{s_1}\right) + \left(\frac{u}{u_1} - \frac{s}{s_1}\right)\left(\frac{\dot{u}_1}{u_1} - \frac{\dot{s}_1}{s_1}\right) = \frac{K_1}{u_1 s_1}. \tag{14.2}$$

To K_2 is to be added the term

$$-\frac{dn_3}{dc_2}\int K_3 dt$$

and

$$\delta n_3 = \text{const. term in } X_\alpha\frac{dX_0}{dc_2} + Y_\alpha\frac{dY_0}{dc_2},$$

with $\delta n_3(u_3/u_1 + s_3/s_1)$ added to the left member of (14.2).

It may be noted that, whatever method of solution of the equations of variations be adopted, the additions to the motions of the perigee and node can be found independently of the coefficients of the periodic terms.

The solution of the equation for z is well known :

$$z = [z_4\int Z_\alpha z_5 dt - z_5\int Z_\alpha z_4 dt] \div C_{54},$$

where

$$C_{54} = \dot{z}_5 z_4 - \dot{z}_4 z_5.$$

To z is to be added the term

$$-z_5\frac{dn_5}{dc_4}\iint Z_\alpha z_5 dt\, dt \div C_{54},$$

and the addition to the motion of the node is given by

$$\delta n_5 = \text{const. term in } Z_\alpha z_4 \div C_{54}.$$

Yale Observatory :
1936 *February* 13.

PHOTOGRAPHIC PHOTOMETRY OF THE ELLIPTICAL NEBULÆ.

R. O. Redman, M.A., Ph.D.

Summary.—Two photographic methods of determining the total apparent brightness of an elliptical nebula are discussed, especially in relation to the systematic error likely to arise from failure to record the outer regions of low surface brightness. The photographic magnitudes and intensity distributions along the principal axes of six nebulæ are given, and the results compared with those obtained elsewhere. A conventional definition of the diameter of an elliptical nebula is suggested.

1. *Introduction.*—For some years past a programme of observation has been pursued at the Solar Physics Observatory, Cambridge, with the object of determining the total apparent luminosities of some of the brighter elliptical nebulæ in the northern sky. Progress has been a good deal slower than had been hoped, partly on account of difficulties to be mentioned in this paper and also on account of frequent and prolonged interruptions caused by other work. Even now the results are very limited, but it is felt that they should be published at this stage, if only to act as some guide to other workers who may be engaged in similar investigations. Since the commencement of the programme, in 1931, the position with regard to the magnitudes of these nebulæ has somewhat improved. Several papers have appeared * giving lists of nebular magnitudes, and that of Shapley and Miss Ames in particular covers practically all the objects on the Cambridge working list. A certain lack of agreement can, however, still be found between the results of different workers, and the present writer remains convinced that too many independent measurements cannot be made in this problem. Bitter experience has shown that these nebulæ provide numerous pitfalls for those wishing to study them photometrically. It appears relatively easy to obtain a self-consistent scheme of measures, but to link these to the stellar magnitude scale without serious error appears to be a matter of much greater difficulty, so that it is with no apologies that the present contribution to the subject is put forward.

2. *Instrument.*—Throughout this work the 3-foot Newtonian reflector at the Solar Physics Observatory has been used. This instrument was built by the late Dr. Common in 1891, and in some respects is a very suitable telescope for nebular photography. For instance, it works at the favourable aperture ratio of 3·8, the linear aperture being 36 inches and the focal length about 136 inches. On the other hand, at the time of commencement of this work it had been lying practically unused for at least twenty years and was consequently in poor condition generally, while little definite information

* *E.g.* Shapley and Ames, *H.A.*, **88**, No. 2, 1932; Seyfert, *H.C.*, 403, 1935; Keenan, *Ap. J.*, **82**, 62, 1935.

was available concerning its optical and mechanical qualities. Not unexpectedly, it has been found that, although a remarkable achievement for an amateur maker working nearly fifty years ago, the telescope is not up to modern optical and mechanical standards. The mounting and principal mirror in 1931 were very much as Common made them, but the Newtonian flat had been replaced in 1926 by a modern mirror which was assumed to be of the best quality. The original driving clock had been replaced by a Gerrish drive which has given very good service.

Work was commenced with the telescope in this state, no modifications being made apart from cleaning and readjusting. It was soon found that the primary mirror possessed a large amount of permanent astigmatism, which was measured by Hartmann tests made outside the telescope, without the secondary. The astigmatism remained largely unaffected by rotating the mirror during these tests, *i.e.* it moved with the glass. Zonal error was comparatively small. The Hartmann criterion, evaluated so as not to smooth out the astigmatism, was 0·7. In view of these results only a very modest standard of definition could be hoped for in practice, but the instrument was considered good enough for the determination of nebular magnitudes.

Last year, 1935, Mr. C. R. Burch very kindly undertook to refigure the primary mirror. At first after its return no very marked improvement in definition was noted. The trouble was traced to the secondary flat, which had never been suspected of possessing any errors comparable with those of the primary since it was made so recently, the observed astigmatism shown by knife-edge tests on stars having always been attributed to the primary alone. Now, however, the secondary was shown to be markedly concave, and it too is at present being refigured by Mr. Burch. Meanwhile an old Common flat which happened to be available is being used, and, with the refigured primary, the definition is now much better than ever before.

3. *First Method of Measurement.*—In view of the general quality of the photographs which could be obtained, the first method of measurement of the nebular images avoided any attempt to determine the real distribution of surface brightness and was as direct as possible. A nebula was photographed in focus, and on the same plate an extrafocal exposure was made of a field of standard stars, for exactly the same length of time and at very nearly the same zenith distance as the nebula. The extrafocal images were normally about 0·6 mm. in diameter. The scale of the plates is almost exactly 1 mm. = 1′. The plate was calibrated by means of a tube sensitometer, the standard patches being superposed on the sky fog. The calibration exposure time was generally of the same length as the telescope exposures; sometimes it was shorter, but rarely by a factor of more than $\frac{1}{3}$ or $\frac{1}{4}$. At times the procedure was varied by using for the standard stars a second plate, taken from the same package as the nebular plate and developed with the latter. In that case the intensity of illumination in the sensitometer was usually varied between the two plates, to allow a fuller range of spots to contribute to the final characteristic curve. The advantage of using a second plate is that it avoids superposing a double dose of sky fog (or a star image) upon the nebula. After calibration the plates were

stored for a period of the order of 24 hours. They were then brush-developed in M.Q. at 19° C., the temperature being thermostatically controlled. The plates were held in a recess so that the emulsion lay flush with a false bottom in the developing dish, the object being to encourage a smooth flow of developer over the surface. Earlier work was done on Ilford Zenith 400 or 650 plates. More recently Eastman 40 plates have been employed, since on the whole they seem to be faster, cleaner and of better grain than the 650 plates. All plates were backed. This is essential for nebular work.

The plates were measured with the recording microphotometer * at the Solar Physics Observatory. In this instrument, constructed primarily for dealing with spectrograms, a reduced image of a lamp filament, a straight closely wound spiral, is projected on the plate. An enlarged image of the illuminated area of the emulsion is projected on a slit behind which is the recording photoelectric cell. The rest of the instrument follows well-known lines and need not be described here. For the nebulæ it was necessary to measure a small circular or square area of the plate, not a thin strip as in spectrophotometry, and the only means of doing this with the machine in its normal state is to limit the used length of lamp filament by means of a suitable diaphragm in the condensing system. By this procedure, and with a wide photometer slit, an area about 0·07 × 0·07 mm. on the plate could be isolated. As will be described later, there is reason to believe that to rely only on limiting the length of lamp filament in this manner is not altogether satisfactory and may lead to error, so that now a circular diaphragm is used over the slit.

Microphotometer tracings were taken across the nebula at intervals of 0·1 mm. and measured at similar intervals, so that effectively the relative intensities within the nebular image were derived for all points at the corners of a network of squares of 0·1 mm. side. A similar procedure was followed with the standard stars, three or four being measured on each plate. Straightforward summation then gave the total luminosity of the nebula and of the stars, so that, the photographic magnitudes of the latter being known, the total light of the nebula could be expressed in stellar magnitudes. During the reduction a correction had to be applied to the measured intensity of each star, according to its position relative to the centre of the field, on account of a certain amount of vignetting. The secondary mirror of the telescope is not quite large enough to transmit a full beam from the primary, and the light loss increases from the centre of the field outwards, in a manner which can be calculated geometrically. (It should be noted that the correction is not symmetrical, with regard to the centre of the field, for displacements parallel to the axis of the telescope tube.)

4. *Errors of the First Method.*—The sources of error of the first method may be classified as follows :—

(i) Under-exposure of the outer regions of the nebula.
(ii) Variable sky.

* Carroll and Moss, *M.N.*, **91**, 191, 1930.

(iii) Inaccurate calibration of the plate or poor photographic procedure.
(iv) Inaccurate measurement of the plate densities.
(v) Irregularities in the plate emulsion.
(vi) Inaccurate magnitudes of the standard stars.

(i), (iii), (iv) and (vi) may lead to systematic as well as accidental errors. The latter are by far the more important since they cannot be reduced by repeating observations without change of procedure. We should observe that the above errors are not by any means peculiar to the method just outlined.

(i) The earlier results derived by Method 1 were encouraging. It is true that the method is tedious and laborious, but it gave magnitude values which agreed from plate to plate in a manner consistent with an average error about $0^m.1$ per exposure. At this stage the writer attended a talk given in Oxford by Dr. Hubble who, in the course of a description of much wider investigations in progress at Mount Wilson, referred to the difficulty arising in the case of the elliptical nebulæ from under-exposure of the outer parts of the image. His estimate of the error likely to arise from this source was greater than $0^m.1$, so that, although it did not seem the Cambridge results could suffer from this trouble to any serious extent, on account of their internal consistency, measures were immediately made of one or two plates of longer exposure than usual. The results were shocking. The total luminosity of N.G.C. 221, for example, immediately jumped up by about $0^m.5$. Later measures, it is true, have shown somewhat less change than this first result, and it has been found that the error becomes greater with increasing brightness of the nebula, so that N.G.C. 221 is one of the worst cases which could have been selected. Nevertheless, Method 1 is clearly not altogether satisfactory. The internal consistency of these earlier measures seems to have been due to the fact that exposure times had been fairly uniform, although not entirely so. It is instructive to see some of the figures actually obtained (Table I).

If direct photographs in focus are to be used, clearly we must increase the earlier exposure times considerably. These exposure times were mostly of the order of half an hour, Ilford plates being used. To make sure of recording all that matters of the nebula it appears necessary to expose, in Cambridge conditions, for at least two or three hours on Eastman 40 plates. Then, however, the central regions are usually heavily over-exposed, and the direct method of measurement again loses in accuracy. The fundamental trouble is that the surface brightness of the nebula varies from a rather high value at the centre down to zero, and that it is essential to measure over a far wider intensity range than can be dealt with photographically in a single exposure. We are led to seek alternatives to Method 1, but, before describing these, reference must be made to the other sources of error.

(ii) Care was exercised always to select a standard star region at very nearly the same zenith distance as the nebula, so that differential absorption taken from a table of (photographic) atmospheric extinctions, such as that in *Handbuch der Astrophysik*, 2, 490, 1931, never exceeded $0^m.05$. The

TABLE I

Magnitudes obtained by Method 1

These magnitudes are given to show the systematic error which is likely to be introduced by under-exposure. Compare the magnitudes of Table III, derived by Method 2, in part from the same plates.

Nebula	Plate	Exp. m	Comparison Stars	Magnitude
N.G.C. 221	N82	19	Harvard Standard Region B1, stars M, O, P, T	9·07
	N83	14	,, ,, ,, ,, ,,	9·14
	N85	12	,, ,, ,, ,, ,,	9·23
	N90	10	,, ,, ,, ,, ,,	9·16
	N91	10	,, ,, ,, ,, ,,	9·42
N.G.C. 2672	N64	60	North Polar Sequence, stars 16, 17, 18	12·96
	N65	60	,, ,, ,, ,, 16, 17, 18	12·54
	N68	60	,, ,, ,, ,, 16, 17, 18, 22	12·63
	N69	60	,, ,, ,, ,, 16, 17, 18	12·79
N.G.C. 2768	N108	32	Harvard Standard Region B4, stars U, W, X, Y	11·59
	N112	40	,, ,, ,, ,, W, X, Y, Z	11·69
	N126	30	,, ,, ,, ,, X, Y, Z, *b*	10·75

final aim was to keep the average error of one magnitude determination below $0^m{\cdot}1$, so that atmospheric absorption within the limits observed should give rise to no trouble. Small corrections taken from the table quoted were applied to the measures.

A much more serious question is that of changes in sky transparency during the two successive exposures. The sky in England can be quite good at its best, but normally it is subject to great variation, and a night of steady transparency throughout is a rarity. In addition there is at Cambridge, situated in flat country not much above sea-level, a marked tendency for ground fog to accumulate gradually after sunset. The most that could be done in this work was to keep a sharp watch on the sky by naked eye during the course of the nebula and star exposures, and, if a definite change of transparency could be seen, to reject the plate for magnitude determination. This has led to a fair amount of wasted time, but no other course seems possible if errors are to be kept below $0^m{\cdot}1$. The consistency of the measures to be described below shows that the precaution just mentioned is sufficient to the degree of accuracy required.

(iii) In the calibration of the plates a number of separate errors may arise. Some of these can be seen without further investigation to be negligible. The timing of nebula and star exposures was accurate to a second, so that any possible differences were entirely unimportant. The sensitometer exposure was sometimes permitted to vary from the corresponding telescope exposure, but by a small factor only. It is fairly certain that the characteristic curve of a photographic plate, particularly when not dyed, varies very slowly with exposure time, and we are therefore quite safe from errors of the order of $0^m{\cdot}1$ arising on this account.

Similarly, we may ignore the fact that the mean wave-length of the light from the nebula is not the same as that of the electric light used for calibration.

The colour indices of these nebulæ are usually taken to be around $1^{m}.0$, so that the mean wave-length which will affect an undyed photographic plate, after transmission through the atmosphere and reflection by two silver surfaces, cannot be more than one or two hundred angstroms from the mean effective wave-length of a tungsten filament after transmission through three diffusing screens. It is well known that undyed photographic plates show only a slow change of characteristic curve with wave-length over the region λλ 4000–5000.

No intermittency effects enter into the question as they do sometimes in spectrographic work, but exposures at the telescope were made in damp, cool air, while the calibration was done indoors in comparatively dry and warm conditions. This difference has been neglected. Changes of the latent image between exposure and development, whether they occur or not, have been taken care of by storing the plates before development, as already described.

An investigation was made into the accuracy of the tube sensitometer itself. In construction this consists of a series of apertures illuminated uniformly through suitable diffusing screens, each aperture transmitting light to the plate by way of a light-tight tube of a fixed length, great relative to the diameter of the aperture. The light intensities affecting the plate are proportional to the aperture areas, which may be measured geometrically with ample accuracy. The chief dangers are from light reflected by the walls of the tube and from non-uniformity in the illumination of the apertures. The first is minimized by the use of suitable diaphragms in each tube, and both were checked by the tests described below.

Two different sensitometers have been used for the work. The first suffered from the defect that the intensity marks ("spots") on the plate were in two rows, both giving a steady change of intensity in the same direction across the plate, an arrangement liable to lead to error in unfavourable circumstances. The second model corrected this. The spots were increased in number from 10 to 12 and were distributed more at random over the plate. With the sensitometer exactly as it was used for calibrating the nebulæ plates, tests were made by comparison with carefully controlled series of exposures (outside the sensitometer) with interchangeable diaphragms, the latter being employed in a manner which excluded all possibility of systematic error. The results were satisfactory, with an average residual 0·02 in log I for spots 2 to 11 inclusive (Table II). The check measures for the first and last spots were too few to be reliable, and the residuals for these were markedly greater. In the case of spot 1, the blackest of all, there appears to be trouble from reflections in the tube. Fortunately both 1 and 12 usually play a very minor part in determining the photographic characteristic curve. There seems to be no systematic trend in the residuals from the other spots, and it has been concluded that the tube sensitometer is quite accurate enough for the work in hand.

(iv) The measurement of the plate densities has been described. We recall that the area being measured at any given moment was originally defined in one co-ordinate by limiting the length of lamp filament projected

TABLE II

Result of Tests of Tube Sensitometer

Spot Number	log *I*		Residual
	From Aperture Areas	From Test Exposures	
1	1·96	2·16	0·20
2	1·78	1·81	·03
3	1·64	1·70	·06
4	1·28	1·31	·03
5	1·19	1·18	·01
6	0·99	1·01	·02
7	0·84	0·82	·02
8	0·65	0·65	·00
9	0·47	0·44	·03
10	0·19	0·18	·01
11	0·07	0·08	·01
12	0·01	0·08	·07

on the plate. This procedure may be satisfactory for the image of the nebula itself, where, apart from the centre, the density gradient is small, but has been suspected of giving rise to inaccuracy in the case of the star images. The condensing system of the photometer contains two single lenses and is neither achromatic nor properly corrected for any other optical aberration. The image of the filament on the plate thus cannot be perfectly sharp, and as a consequence the resolving power of the microphotometer, when used in the manner described, is reduced and the extrafocal star images are measured as rather greater in extent than they really are. Using a circular diaphragm in contact with the photometer slit (which is then not used) and taking particular care near the edges of the images, the total light in the stars on remeasurement of a number of plates was reduced by 7 per cent. Any of the earlier star measurements used in this paper have therefore been corrected by this amount. It should be remembered that in any case the measurement of extrafocal images obtained with a Newtonian reflector working at f. 3·8, without a correcting lens, cannot be as accurate as if similar photographs had been taken with a good refractor. Not only does the shadow of the secondary mirror and its supports cut up the image, but coma introduces troublesome distortion as well. If the average error in measuring the summed luminosity of four stars by this method is definitely less than 10 per cent. per plate, we must be satisfied.

It is not profitable to try to measure an area of a fast plate less than 0·07 × 0·07 mm., on account of the irregularity from plate grain. Under these circumstances any structure of diameter not greater than 0′·07 will certainly be distorted in the Cambridge measures. The outer parts of any of the nebulæ measured will be unaffected, but the very sharp peak in the surface brightness distribution, which is to be found at the centres of some, if not all, elliptical nebulæ, will be smoothed over. Fortunately this central region, although of very high surface luminosity, is of relatively very small

area, and its contribution to the total light of the nebula is thus not very great.

(v) Irregularities in the photographic emulsion in general merely contribute to the accidental error of the final result. Ordinary plate grain is a great nuisance, as always, and in the direct method outlined above it not infrequently conceals all sign of the nebula on tracings which cross only the faint outer regions. Variations of plate sensitivity from point to point, as distinct from ordinary grain, may also prove a difficulty, since they can in bad cases lead to fluctuations in the background, of the same order as the low densities which must be measured in the outer regions of the nebula. The usual practice was to reduce all readings to percentages of the difference "clear film" (actually the background sky fog) to "total blackness" and work with these whatever the fluctuations of the "clear film" reading. It seems impossible always to avoid these background irregularities, whose source is as likely to be in the processes of manufacture as much as in subsequent treatment of the plate.

Eberhard effect should not create any trouble, especially in view of the care with which the plates have been developed. It may possibly push up the apparent central intensity of the nebula, but it is doubtful whether at most it does more than compensate for a small part of the depression of the central intensity by finite resolving power of the telescope, the plate and the microphotometer. At all events the effect has been regarded as negligible.

(vi) The stars used for this work were generally those of the Harvard Standard Regions. Unfortunately the original magnitudes published in *H.A.*, **71**, No. 4, 1917, seem to contain a rather large scale error. Corrections have been published by Miss Payne in *H.A.*, **89**, No. 1, 1931, but these do not cover stars beyond about $11^{m}.0$, while for the nebulæ it was sometimes necessary to go as faint as 14^{m}. It seems risky to extrapolate the corrections, so that the original magnitudes have been used. Presumably accurate magnitudes will in due course become available for these stars, and it will then be a simple matter to correct the figures for the nebulæ, which are based on direct comparison with the summed brightness of the three or four comparison stars.

5. *Second Method of Measurement.*—The primary difficulty in this work is the fact that these nebulæ shade off indefinitely from the centre outwards, with no clear-cut boundary anywhere. The outer parts are of very low surface brightness but relatively great area, and are thus very difficult to measure while yet contributing a considerable fraction of the total luminosity. The behaviour of the photographic plate with regard to these very low surface brightnesses has not always been clearly described in astronomical publications. It is not entirely a matter of getting over a certain "threshold," fixed for a given kind of radiation and exposure time, since every plate taken, with the exception only of quite short exposures, has a perceptible amount of sky fog on it. The light intensity at any point of the nebula, however small it may be, when added to this fog must be above any plate threshold, which is in general of less importance than the irregularities of

the plate emulsion. Fundamentally all measures of photographic density are statistical. A certain amount of a given kind of radiation, say P, when incident on the plate under given circumstances will produce a certain quantity Q of developable silver grains. Q is subject to random fluctuations of an average amount, δQ say, depending largely on the type of plate. We work on the basis of the fact that there is a relation between P and the average $\bar{Q}$, a relation which is expressed in the characteristic curve of the plate. The fundamental problem is to measure small differences of radiation intensity, and by confining attention to equal areas of plate this, in practice, becomes the measurement of small differences in amounts of radiation, viz. $P_1 - P_2$, where P_1 arises from nebula plus sky fog and P_2 from sky fog alone. As we pass from the centre of the nebula outwards $P_1 - P_2$ tends towards zero, and there comes a point where the corresponding $\bar{Q}_1 - \bar{Q}_2$ is of the same order as δQ; in other words, where accidental error becomes of the same size as the quantity to be measured. It is clear that at this stage $P_1 - P_2$ becomes indeterminate. In dealing with a nebular image we may effect some improvement by increasing the area of plate being measured in the microphotometer at any given moment, thus increasing Q relative to δQ, but with a finite nebular image the possibilities in this direction are quite limited. We may also arrive at some improvement by increasing exposure times so as to work on the greatest slope of the characteristic curve of the plate, but it is to be noted that an indefinite increase of exposure time in the presence of sky fog does not bring about an indefinite increase of ability to measure $P_1 - P_2$, for $Q_1 - Q_2$ commences to decrease after a time, although individually Q_1 and Q_2 may still be increasing. On a good night at Cambridge, with Eastman 40 plates and the f. 3·8 reflector, little or no improvement seems to be attained by exposing more than about three hours, although this opinion has not been tested by any rigorous measurement. Somewhat better results seem to be obtained with Ilford Hypersensitive Panchromatic plates, using about the same exposure time, but this may perhaps be attributed to their somewhat greater contrastiness as much as to their extra speed. The results quoted a little later are from plates, including some taken with about the optimum exposure time just suggested, and it is doubtful whether much greater extension of these nebulæ can be obtained at Cambridge, except by photographing on unusually transparent nights such as occur only two or three times a year.

To get over the difficulty provided by the outer regions of the nebula, recourse may be had to other methods, such as extrafocal photography, but these are open to criticism for reasons to be explained later. The only certain and accurate photographic method yet proposed is that in which the distribution of light within the nebula is carefully determined from a number of different photographs and an integration then performed. This method, which we will call Method 2, has been used by Hubble on a few objects, although there is reason to believe that he has not pushed his measures out to sufficiently low surface brightnesses. For statistical work Method 2 is impossible on account of its slowness, but there seems no reason why it should not in time be applied to some scores or few hundreds of the

brighter nebulæ and to a selected few of the fainter ones, when it should act as a very necessary check on the accuracy of quick methods designed chiefly for statistical work. Method 2 is not quite the same as Method 1, where it was attempted to determine the total luminosity from each single exposure. Using focal images, this cannot be done with any accuracy owing to the fact that the surface brightness which must be recorded may extend over a range of as much as 10,000 to 1, whereas the normal photographic plate certainly does not deal adequately with a range greater than 100 to 1. We have then to work with a number of different exposures, a part of the intensity distribution being determined from each, and then a composite picture of the nebula built up.

It is generally supposed, from the work of Hubble and others, that the isophotal contours in these nebulæ are approximately ellipses, and in consequence it seems sufficient if the light distribution along the major and minor axes is determined, or along any two perpendicular axes if the nebula is circular. The whole luminosity is then found on the assumption of elliptical isophotal curves. The ellipses need not be similar; indeed the Cambridge measures indicate that in fact they are not. We take a series of convenient intensities I_1, I_2, I_3, . . . and find the points on the axes where these values are attained. Suppose the points for I_n are A_n', A_n'' on the major axis, and B_n', B_n'' on the minor axis. Then $A_n'A_n'' = \alpha_n$ is the major axis of an ellipse, and $B_n'B_n'' = \beta_n$ its minor axis, and the total luminosity of the nebula is taken to be

$$\sum_n \pi(\alpha_{n+1}\beta_{n+1} - \alpha_n\beta_n)\tfrac{1}{2}(I_{n+1} + I_n).$$

To test whether we have measured far enough out along the axes, the contributions from the lower intensities are considered separately. Suppose i is the lowest intensity measured in the nebular image. Then plot against intensity the contributions from the intensity intervals i to $2i$, $2i$ to $3i$, $3i$ to $4i$, etc. They form a curve, generally rising with diminishing intensity, and we have to extrapolate this curve from $\frac{3}{2}i$ towards zero intensity so as to obtain the ordinate at $\frac{1}{2}i$, which will give us the contribution from the intensity interval 0 to i. If the measures have been pushed out far enough it will be found that any reasonable extrapolation of the curve will add only a small percentage (in good cases less than 5 per cent.) to the total luminosity. In such circumstances we may feel confident that, after adding the small extrapolated contribution, no serious error is being made by reason of the fact that the extremely tenuous outermost regions of the nebula have not been recorded.

The intensity scale, as far as distribution within the nebula is concerned, is a purely relative one and it has finally to be converted into an absolute scale, viz. stellar magnitudes. For this, standard stars are used, as in Method 1, but now we need not attempt to record the whole of the nebula on the standard star plates. It is necessary only to measure a convenient part of the nebular image, which need not even include the centre, and to relate this to the star intensities. From the complete distribution figures we can readily find what fraction of the whole nebula we are measuring, and

hence can deduce the total luminosity in stellar magnitudes. Results derived in this way for five nebulæ are to be found in Table III, which should be compared with Table I.

TABLE III

Photographic Magnitudes of Six Nebulæ, obtained by Method 2

The total magnitude relies on the distribution data of Table IV. In each case the latter have been extrapolated in the manner described in § 5, the fraction of the whole luminosity coming from the extrapolation being tabulated. Note that the magnitudes are brighter than those of Table I, while individual plates agree better with each other. The H.S.R. magnitudes are from *H.A.*, **71**, No. 4, 1917, and are systematically too bright relative to the international photographic scale. Full corrections are not yet available ; they may be of the order $0^m{\cdot}1$ to $0^m{\cdot}3$ (*H.A.*, **89**, No. 1, 1931), increasing with decreasing star luminosity. The errors of the nebular magnitudes depend also on the errors of the distribution measures.

Nebula	Plate	Comparison Stars	Extrapolated Fraction	Magnitude
N.G.C. 221	N82	H.S.R. B1, M, O, P, T	0·021	8·75
	N83	,, ,,		8·77
	N90	,, ,,		8·55
	N91	,, ,,		8·74
2672	N64	N.P.S., 16, 17, 18	0·077	12·80 *
	N65	,, 16, 17, 18		12·63
	N68	,, 16, 17, 18, 22		12·68
	N69	,, 16, 17, 18		12·67
2768	N108	H.S.R. B4, U, W, X, Y	0·028	11·03
	N112	,, W, X, Y, Z		11·08
	N126	,, X, Y, Z, *b*		10·94
3607	N132	H.S.R. C6, *b*, *d*, *e*, *g*	0·041	11·01
3608			0·070	11·63
3605			0·028	13·25

The mean magnitudes, together with values from *H.A.*, **88**, No. 2, 1932, and the Holetschek (visual) figures, revised by Hopmann (*A.N.*, **214**, 425, 1921) are as follows :—

Nebula	Cambridge	*H.A.*, **88**, No. 2	Holetschek (Visual)
N.G.C. 221	8·70	9·5	8·8
2672	12·68	12·6	12·8
2768	11·02	12·0	10·7
3605	13·25	13·1	12·5
3607	11·01	11·4	9·9
3608	11·63	12·5	11·6

Other photographic magnitudes of N.G.C. 221 are 8·7 (Shapley, *Proc. Nat. Acad. Sci.*, **15**, 569, 1929) and 9·6 (Hubble, *Ap. J.*, **74**, 55, 1931).

Method 2 has at least three advantages over Method 1, viz. : (*a*) It deals more adequately with the outer regions, and during the reduction of the measures it is possible to make an estimate of the error likely to be made

* An observing note says : "Signs of improved transparency during second exposure," *i.e.* during standard star exposure. Observation given half weight.

on account of neglect of the outermost parts. (*b*) It does not need such long exposures for the comparison with the standard stars, a very important point if work is being done in a poor climate. (*c*) It does not need so many microphotometer tracings.

6. *Other Methods.*—There are several other methods which it is possible to employ for determining the total apparent luminosity of these nebulæ. Two of these aim at making the images of both stars and nebulæ big and of uniform brightness, so that a measure of the blackening of the plate within the image can give the total brightness of the corresponding star or nebula. The procedure is either to use extrafocal images or to employ some apparatus such as the Schraffierkasette. To the writer it seems doubtful whether either device can be made to give very satisfactory results with elliptical nebulæ brighter than, say, 13^m, simply on account of their extension. The Cambridge measures show that a 9th or 10th magnitude nebula may extend to a diameter of at least 10′, while it has been possible to trace 11th or 12th magnitude objects to diameters 3′ or 4′. With such nebulæ any method of measurement has therefore to deal with all the light coming from a quite considerable area of sky. Nebular images obtained extrafocally or with the Schraffierkasette will not be sufficiently uniform to give reliable comparison with stars until they exceed the true nebular diameters by a considerable amount. Even if the enlarged images are measured by a process similar to that of Method 1 they would have to be two or three times the "diameter" of the nebula (as quoted above) before they showed any marked advantage over focal images. It would seem that images of at least 10′ or 20′ diameter, sufficiently strongly exposed to show the desired objects, would suffer very much indeed from overlapping. For *faint* nebulæ, however, this objection does not apply to anything like the same extent, and the methods just mentioned might then be much more useful.

Similar trouble is experienced with another method, actually tried at Cambridge, which utilizes the photographic analogue of the Maxwellian field of view, following a principle proposed first by Fabry. A small lens is placed in the focal plane of the telescope, and with it the telescope objective is photographed. If the lens is situated so that its axis coincides with that of the telescope, it will photograph the objective in the light of any celestial object which has been brought to the centre of the field. Since the objective is uniformly illuminated by the star or nebula, as the case may be, the result is a uniform patch on the plate, which is particularly easy to measure photometrically. It is easy to devise means of dealing with the superposed sky fog. The difficulties are, first, that any stars or other nebulæ within the borders of the nebula being measured will also contribute to the plate blackening, and, secondly, that the ratio aperture/focal-length of the lens has to be very big, for the linear aperture has to be great enough to include all the nebula, while the focal length has to be short enough to give reasonably bright illumination on the plate. If the nebula is small enough, the method is a promising one, particularly with a Newtonian reflector where the ordinary extrafocal images are not very good for photometry, but it cannot readily deal with more than one nebula per exposure.

Another method, used extensively at Harvard, employs photographs of nebular fields taken with a short focus lens. The nebular images are then reduced to very small proportions on the plate and are directly compared with star images. Although the scale of the plates may be very small, the images of the nebulæ may still be of considerable size. For instance, on a scale 1 mm. = 10′, which is approximately that of many of the Harvard plates, the image of a 12th magnitude nebula will probably be about 0·3 mm. in diameter, presumably considerably bigger than star images in focus. While useful for a preliminary survey, we may fear that this method may therefore give systematic errors, depending on the magnitude of the nebula and possibly also on its shape. There is some evidence for this in the magnitudes of *H.A.*, 88, No. 2, when compared with the Cambridge values, and the recent work of Seyfert * goes to show that for 9th or 10th magnitude nebulæ the small scale method is unreliable.

7. *Distribution of Light along the Nebular Axes.*—As a product of Method 2 we obtain, besides the magnitude, the light distribution along the axes of each nebula. From a physical point of view this is at least as important as the total luminosity, but until the refiguring of the 3-foot mirror it seemed inadvisable to attempt distribution measures. With the improved definition now available, satisfactorily accurate data can be secured for regions more than 0′·1 from the centre. The results for the six nebulæ dealt with in this paper are in Table IV, which gives the logarithms of the intensities, adjusted so that $\log I = 2$ at the centre.† We must remember that the central value is certainly too low in every case and that the error may be expected to increase with diminishing nebular size and luminosity, on account of inadequate resolving power. For this reason we must expect the lowest intensity recorded to become progressively greater relative to the maximum central intensity as we pass from brighter to fainter nebulæ.

The first point to notice about the figures of Table IV is the much greater extent of the nebulæ than would be expected on the basis of ideas prevalent a few years ago (see also Table V). Stebbins's photoelectric measures and recent photographic work at Harvard have prepared us for this, but the results are still quite striking. Table VI gives the total luminosity of each nebula out to surface brightnesses 0·1, 0·01 and 0·001 of the observed value at the centre, and allows some idea to be obtained as to how far it is necessary to go in order to record the major part of the total light of the nebula.

Hubble's measures of the distribution of light within elliptical nebulæ ‡ include only one of the objects of Table IV, namely, N.G.C. 221. The

* *Loc. cit.*

† As noted in Table IV, panchromatic plates have been used sometimes for the outermost portions of the nebulæ. It has been assumed that the distribution in these regions is the same for wave-lengths $\lambda\lambda$ 5000–6500 A. as for $\lambda\lambda$ 4000–5000 A., the wave-lengths mostly used by ordinary undyed photographic plates. There is no reason to suppose this to be seriously at variance with the facts, since the distribution relatively near the centre has been shown by Mount Wilson work to be independent of wave-length.

‡ *Ap. J.*, 71, 231, 1930.

agreement between the Mount Wilson and Cambridge measures is very satisfactory out to a distance about 1′·5 from the centre, but from that point onwards the Cambridge values become progressively lower than the Mount Wilson figures. Curiously enough, there is agreement right up to the centre, where one would have expected the superior instrumental and atmospheric conditions at Mount Wilson to give a sharper peak in the distribution. The systematic difference beyond radius 1′·5 appears where Hubble's measures begin to rely on a few long exposures. Hubble makes it clear, however, that the photometry of these plates is less reliable than that of the shorter exposures. It should be noted also that the formula with which the light distribution is expressed in his paper gives an infinitely great luminosity when extrapolated and integrated to infinity. At some point, then, the surface brightness gradient must in reality be greater than that of the formula, a conclusion which is confirmed by the Cambridge work.

The figures of Table IV show that the ratio of the axes in both N.G.C. 221 and 2768 varies with distance from the centre. Both nebulæ tend to become more circular at very great radii. On the other hand, the central regions of N.G.C. 2768 seem less elongated than intermediate parts. It may well be that the behaviour of 2768 in this respect is typical of the highly elongated nebulæ.

8. "*Diameters.*"—Enough has been said to make it clear how extremely difficult it is to discuss questions relating to the diameters or mean surface brightnesses of these objects. All diameters published so far are as much functions of the instrumental and atmospheric conditions under which the photographs were made as of nebular structure. There is in fact no boundary to any of these nebulæ; they all shade off imperceptibly into

TABLE IV

Distribution of Luminosity along Major and Minor Axes

The values are in logarithms to the base 10 and are adjusted so that log $I=2$ at the centre. Distances from centre are in minutes of arc.

1. N.G.C. 221. Type E2:

Distance from Centre	Major Axis	Minor Axis	Distance from Centre	Major Axis	Minor Axis
′			′		
0·0	2·00	2·00	1·8	$\bar{1}$·30	$\bar{1}$·06
0·1	1·50	1·33	2·0	·15	$\bar{2}$·93
0·2	·01	0·92	2·4	$\bar{2}$·88	·72
0·4	0·58	·46	2·8	·65	·53
0·6	·35	·17	3·2	·48	·37
0·8	·13	$\bar{1}$·92	3·6	·33	·24
1·0	$\bar{1}$·94	·71	4·0	·21	·14
1·2	·77	·52	4·4	·11	·05
1·4	·60	·35	4·8	·02	$\bar{3}$·96
1·6	·45	·19	5·2	$\bar{3}$·98	·88

Number of plates, 14. Exposures 10^s to 175^m. Two of the long exposures are on Ilford Hypersensitive Panchromatic plates.

2. N.G.C. 2672. Type E2 :

Distance from Centre	Major Axis	Minor Axis	Distance from Centre	Major Axis	Minor Axis
′			′		
0·0	2·00	2·00	0·8	$\bar{1}$·98	0·00
0·1	1·72	1·59	1·0	·72	$\bar{1}$·72
0·2	·32	·21	1·2	·52	·50
0·4	0·75	0·70	1·4	·34	·30
0·6	·30	·32	1·6	·20	·14

Number of plates, 7. Exposures 60^m to 220^m. The shorter exposures are on a relatively slow plate, Ilford Zenith 400, taken before the mirror was refigured. Better results would be obtained near the centre, using the refigured mirror and faster plates, with shorter exposures. Owing to the superposition of a star and another nebula, the axes have been taken to lie along the line joining nebula centre to star ("minor") and perpendicular to this.

3. N.G.C. 2768. Type E6 :—

Distance from Centre	Major Axis	Minor Axis	Distance from Centre	Major Axis	Minor Axis
′			′		
0·0	2·00	2·00	2·0	$\bar{1}$·76	$\bar{1}$·08
0·1	1·72	1·53	2·4	·56	$\bar{2}$·92
0·2	·42	·06	2·8	·38	·78
0·4	0·98	0·56	3·2	·21	·66
0·6	·66	·20	3·6	·06	·56
0·8	·45	$\bar{1}$·95	4·0	$\bar{2}$·91	·45
1·0	·28	·75	4·4	·78	·36
1·2	·16	·58	4·8	·66	·27
1·4	·05	·42	5·2	·55	...
1·6	$\bar{1}$·95	·28	5·6	·45	...
1·8	·85	·18			

Number of plates, 8. Exposures 30^m to 164^m. The longest exposure is on an Ilford Hypersensitive Panchromatic plate, the shortest on Ilford Zenith 400, the latter taken before the refiguring of the mirror. The central results can probably be improved, as for N.G.C. 2672.

4. N.G.C. 3605. Type E4 :

Distance from Centre	Major Axis	Minor Axis	Distance from Centre	Major Axis	Minor Axis
′			′		
0·0	2·00	2·00	0·4	0·78	0·37
0·1	1·70	1·59	0·6	0·04	$\bar{1}$·68
0·2	1·48	1·12			

Number of plates, 6. Exposures 15^m to 180^m.

5. N.G.C. 3607. Type Pec :

Distance from Centre	Major Axis	Minor Axis	Distance from Centre	Major Axis	Minor Axis
′			′		
0·0	2·00	2·00	1·0	0·22	0·00
0·1	1·67	1·60	1·2	$\bar{1}$·99	$\bar{1}$·76
0·2	·38	·24	1·4	·77	·52
0·4	0·98	0·86	1·6	·55	...
0·6	·70	·56	1·8	·35	...
0·8	·46	·26			

Number of plates, 8. Exposures 5^m to 180^m.

6. N.G.C. 3608. Type E2 :

Distance from Centre	Major Axis	Minor Axis	Distance from Centre	Major Axis	Minor Axis
′			′		
0·0	2·00	2·00	1·2	$\bar{1}$·82	$\bar{1}$·62
0·1	1·54	1·49	1·4	·61	·40
0·2	·25	·12	1·6	·42	·22
0·4	0·86	0·70	1·8	·25	·04
0·6	·56	·39	2·0	·10	$\bar{2}$·88
0·8	·28	·11	2·2	$\bar{2}$·95	...
1·0	·04	$\bar{1}$·86			

Number of plates, 8. Exposures 5^m to 180^m.

Table V

Measured Angular Diameters

The diameters here tabulated are in minutes of arc. The values show how complete is their dependence on observing conditions, owing to the non-existence of any definite boundary. Hubble's values are from comparatively early work (*Ap. J.*, 64, 331, 1926) ; Shapley's are more recent (*H.A.*, 88, No. 4, 1934).

N.G.C.	Cambridge		Shapley		Hubble
	Major	Minor	Major	Minor	
221	10·4	10·4	3·8	3·0	2·6
2672	3·2	3·2	...	...	0·4
2768	11·2	9·6	...	...	1·5
3605	1·4	1·2	...	...	0·3
3607	4·6	4·0	2·7	2·3	1·3
3608	3·6	3·1	2·3	2·3	0·6

Table VI

Percentages of the total luminosity of nebulæ included in regions of surface brightness greater than 0·1, 0·01 and 0·001 respectively of the observed surface brightness at the centre.

Nebula	Magnitude	Percentages		
		Above 0·1	Above 0·01	Above 0·001
N.G.C. 221	8·7	19	59	86
2672	12·7	44	78	94
2768	11·0	26	59	86
3605	13·3	66	94	98
3607	11·0	29	77	95
3608	11·6	36	82	96

empty space, so that we can only speak of a "diameter" by defining some conventional boundary, much as we have a conventional definition for the "half-width" of a spectrum line, another entity which is not definitely bounded. If at any time it becomes possible to measure the surface brightness at the centre of an elliptical nebula without serious error, we shall be able to define the conventional diameter as given by points where

the intensity is some given fraction of the central intensity, but as long as the present state of affairs persists, and the measured central intensity is purely instrumental, this course is not going to lead to any useful result. The best alternative is to define the boundary as the locus of points at which the surface brightness has a certain value in stellar magnitudes per square second of arc. The least surface brightnesses measured on the five brighter nebulæ of this paper are all roughly of the same order, the value for N.G.C. 221 being brightest, probably because it is superposed on the outer parts of N.G.C. 224, the Great Nebula in Andromeda, a circumstance in effect increasing the sky fog. Omitting N.G.C. 221 as an exceptional case, the mean limit is $25^{m}\cdot3$/square second of arc, which is practically identical with the limit $25^{m}\cdot2$/sec.^{2} found by Hubble * for the faintest surface brightness detectible on certain photographs (70–90 minutes exposure) taken with the 60-inch reflector at Mount Wilson. Hubble concludes that a reasonable limit for prolonged exposures under the best conditions is probably between $26^{m}\cdot5$/sec.^{2} and $27^{m}\cdot0$/sec.^{2}. This applies to Mount Wilson; the Cambridge limit would not be quite so faint. It seems that a conventional boundary at 25^{m}/square second of arc would have the following advantages: (*a*) It would allow a definite measure of the "diameter" of the nebula; (*b*) it can be attained on photographs made in an indifferent climate; (*c*) it would include most of the light of the nebula.

The writer must thank Mr. E. G. Shirley, who has carried through much of the computation and measuring used in this paper.

THE CONTINUOUS SPECTRA OF CERTAIN PLANETARY NEBULÆ: A PHOTOMETRIC STUDY.

T. L. Page.

(Communicated by the Director of the University Observatory, Oxford)

Introduction.—The work of Slipher † and Hubble ‡ has shown that the spectra of certain gaseous nebulæ, which are merely illuminated by stars of type later than B1, are completely continuous. Very little attention, however, has been paid to the continuous emission of about the same intensity emitted by the bright line nebulæ. It has been neglected, no doubt, because the much more concentrated light in line spectra needs a short exposure time which would be insufficient to reveal continuous spectrum spread out over all wave-lengths. In the slitless spectra taken by Wright,§ however, continuous emission is noticeable, especially in the

* *Ap. J.*, **76**, 116, 1932.
† V. M. Slipher, *Lowell Obs. Bull.*, No. 55, 1912.
‡ E. Hubble, *Ap. J.*, **56**, 162, 1922.
§ W. H. Wright, *Pub. Lick Obs.*, **13**, 193, 1918, plates xlvi and xlviii.

planetary nebulæ N.G.C. 7662 and N.G.C. 6818, and the combination of continuous and line spectra has also been observed in the diffuse nebulæ N.G.C. 1976 and I.C. 5146.*

It is the purpose of the present contribution to ascertain the origin of this continuous spectrum observed in emission-line nebulæ. The investigation is based on wide slit spectra of five planetary nebulæ taken by H. H. Plaskett at Victoria in 1930, and is concerned in part with the photometric determination of the relative energy distribution in continuous and line spectra between λ 3400 and λ 4900, and in part with the theoretical interpretation of these measurements of energy. *Part I* deals with the method of observation and reduction, *Part II* contains the resulting intensities for the five nebulæ concerned, *Part III* discusses the interpretation of the measured intensity distribution in the Balmer continuous spectrum, and *Part IV* is concerned with the probable origin of the visual continuous spectrum. A *Summary* concludes the paper.

Part I—Spectrophotometry

(*a*) *Method of Observation.*—H. H. Plaskett describes the observational methods connected with the wide-slit spectra in the following terms :—

"The purpose of this observational investigation, carried out during the summer months of 1930, was a determination of the intensity distribution in the continuous spectra of one or more planetary nebulæ. The observational work therefore fell into three parts—the spectrum photography of the nebulæ, the calibration of the plates, and the standardization. These will be described in turn.

"*Nebular Spectra.*—The spectra were photographed with the Hilger Ultra-violet Glass Spectrograph † at the principal focus of the 72-inch telescope of the Dominion Astrophysical Observatory, Victoria. A Zeiss ultra-violet achromat was used in place of the Hilger singlet ultra-violet crown lens supplied with the instrument ; as a consequence the field was sufficiently flat to permit the use of plates over the whole region λλ 3300–5000, while the dispersion was reduced from 52 A. per mm. at λ 3600 to 78 A. per mm. The spectra were taken on Wellington Express Plates (backed before using with 'Eastman Opaque'), this emulsion being chosen on account of its high speed. The spectroscope was oriented on the telescope so that the slit was east and west, and the jaws were opened to 0·5 mm. exactly (approximately 11″ of arc). This meant that a large part of the nebula was included in the slit opening, so that the guiding, by reflection off the polished jaws, was made difficult ; nevertheless it proved possible, when the seeing was good, to obtain practically slitless spectra in good definition. The exposure time varied from 46 to 100 minutes, exposures which resulted in 'blacking out' the line images, but gave a moderately well-exposed continuous spectrum.

"*Calibration.*—In order to calibrate the nebular spectra thus obtained,

* H. Zanstra, *Ap. J.*, **65**, 68, 1927.

† H. H. Plaskett, *Pub. D.A.O.* (Victoria), **4**, 121, 1928.

the Hilger U.V. Spectrograph was removed from the telescope and mounted in a stand in a darkened room. Some 17 cm. in front of the slit was mounted a block of magnesium carbonate with its face (12×9 cm.^{2}) making an angle of 45° with the optical axis of the spectroscope. The centre of the face was not only on the optical axis of the spectroscope but also on the axis of an optical bench 1·7 m. long which carried a six-volt spiral-filament lamp. This lamp, after seasoning, was mounted in a ventilated lamphouse, on the exit window of which could be placed a filter. The distance of the lamp from the block of $MgCO_3$ could be varied between 20 and 90 cm. The calibration exposures, of exactly the same duration as the nebular exposure, were made with the 'neutral' tint wedge, W. iv, placed in contact with the spectroscope slit, and illuminated by the lamp on the optical bench after scattering by the block of $MgCO_3$. Since the transmission of the wedge was much smaller in the violet than in the blue, it was necessary to put two calibration exposures on each nebular plate, one for the region λλ 3900–5100 with the lamp some 100 cm. from the diffusing block and a Corning G50 Signal Blue Filter before the lamphouse, and one for the region λλ 3300–4000 with the lamp distant some 20 cm. and a Corning Heat Resisting Red Purple Ultra Filter (transmission λλ 3150–4000 with a maximum transmission of 65 per cent. at λ 3650). In order to identify wave-lengths in these two calibration exposures, the $MgCO_3$ block was also illuminated for five minutes of the calibration exposure by a filtered quartz mercury arc lamp. These two calibration exposures on each nebular plate, given the wedge transmission as a function of height and wave-length, thus yield characteristic curves of the emulsion for any desired wave-length in the range λλ 3300–5000.

"To determine the wedge transmission (the so-called 'wedge constant') especially in the region λλ 3300–3900, the same arrangement was used as for the calibration exposures, except that the current through the lamp was carefully controlled by two resistances in parallel and an ammeter reading to 0·01 ampere. In order to vary the flux by a given amount, the collimator diaphragm method, elsewhere described,* was used, and with this method 39 plates were taken, yielding 78 diaphragmed spectra to be photometrically matched with 39 wedge spectra.

"*Standardization.*—To standardize these nebular spectra a six-volt spiral-filament lamp, supplied through the courtesy of Dr. L. S. Ornstein, and calibrated at the Physical Laboratory of the University of Utrecht, was used. Colour temperature determinations were made at Utrecht for currents between 4 and 6·5 amperes, the corresponding colour temperatures being about 2200° K and 3100° K. For this particular lamp, partly no doubt on account of the radiation reflected off the coils of the spiral, the filament was found to radiate like a grey body over the wave-length range λλ 3400–6700. Consequently in this range the relative energy distribution of the standard lamp can be found from Planck's formula, using for temperature the 'colour temperature' corresponding to the current at which the lamp was burned.

* H. H. Plaskett, *M.N.*, **91**, 877, 1931.

"The standardizing exposures were made with the spectroscope set up in the darkened room as described for the calibration. The Utrecht Standard Lamp was placed in the lamphouse *without* any filter, and was connected to a storage cell through two resistances in parallel and the Weston D.C. Ammeter (calibrated for this purpose by the Physics Department of the University of British Columbia). The standard lamp therefore illuminated the block of $MgCO_3$ with a flux of known energy distribution. With the same slit width as used for the nebular spectra and the same exposure time, two spectra of the standard lamp, after reflection off the diffusing block, were made—the one with the lamp distant some 180 cm. and a current of four amperes, and the other at the same distance but a current of five amperes in order to reach as far into the ultra-violet as possible.

"With this method of standardization there is peculiar to the Utrecht Standard Lamp the reflectivity of the block of $MgCO_3$, and peculiar to the nebulæ the transmission of the Earth's atmosphere and the reflectivity of the 72-inch telescope mirror. In order to eliminate these two reflectivities two sets of exposures were made, one at the end of July and the other at the beginning of September. On a day with heavy uniform clouds the telescope was turned nearly horizontal and a white sheet (some 2 m. square) was hung about two metres beyond the end of the telescope tube. This sheet was illuminated by diffuse daylight, and the diffuse light from the sheet was reflected by the telescope mirror into the Hilger U.V. Spectroscope mounted in the focal plane of the telescope. One spectrum was taken in this way ; then the spectroscope was removed from the telescope, placed on its stand with the block of $MgCO_3$ in position, and a second spectrum of the diffusely illuminated sheet was taken after reflection by the block of $MgCO_3$. Finally the spectroscope was replaced on the telescope, and a third exposure was made on the same plate after reflection again by the 72-inch mirror. Four plates of identical exposure for each spectrum, the block of $MgCO_3$ in one case bracketed by the mirror exposures, and *vice versa*, were made on July 28 ; and subsequently four more were made on August 30. Each of these plates was subsequently calibrated by wedge exposures of the same duration in the darkened room, using the optical bench as described above.

"This particular method of standardization, while it lacks the simplicity of using a standard star, has the advantage that it permits the determination of energy distribution to the violet of λ 3900—a region in which until the recent work of Chalonge * no accurate determinations of stellar energy distribution had been made. It also has the advantage over the earlier methods of standardization used at Victoria † of taking into account the reflectivity of the whole mirror surface (instead of an isolated patch some 7 cm. in diameter), and of making possible by the use of the optical bench an exact identity of exposure times between unknown and standard sources. As against these advantages is to be set the disadvantage that the flux of daylight reaching the white sheet is not constant either in quantity or quality ;

* Barbier, Chalonge, and Vassey, *Journal de Phys. et le Radium*, **6**, 137, 1935.

† H. H. Plaskett, *Pub. D.A.O.* (Victoria), **2**, 244, 1923.

although systematic errors due to this variability with time are eliminated by the method of bracketing the exposures, the measurement of these reflectivities by Mr. Page has made it clear that the use of some source other than daylight would have reduced the accidental errors.

"*Observational Details.*—With these methods, and after some preliminary trials between July 10 and 22, the nebular plates listed in Table I were taken. The first column contains the plate number, an asterisk indicating that the plate was ultimately measured, the second the exposure time of each spectrum appearing on that plate, while the third main column gives the details of the nebular exposures ; the first entry in this third main column

TABLE I

Observational Data

Plate	Exp. Time	Nebular Exposures				Calibration, U.T.	Standardization, U.T.
		Nebula	*H*	Exposure, U.T.	Def.		
	min.		h m				
W532	72	N.G.C. 7662	3 35 E.	Jul. 24·331	0	Aug. 2·665	Jul. 31·984
						Aug. 2·723	Aug. 1·042
W535	100	N.G.C. 7662	4 00 E.	Jul. 25·310	0	Aug. 4·761	Jul. 31·826
						Aug. 2·789	Jul. 31·905
W536	46	N.G.C. 7662	2 32 E.	Jul. 25·372	0	Aug. 2·039	Jul. 31·019
						Aug. 2·582	Jul. 31·057
W541	80	I.C. 4593	2 22 W.	Aug. 5·244	1	Aug. 7·774	
		N.G.C. 7027	0 59 E.	Aug. 5·311		Aug. 7·835	
		I.C. 4997	1 32 W.	Aug. 5·380			
		N.G.C. 7662	0 03 E.	Aug. 5·445			
W542	80	N.G.C. 6572	0 22 W.	Aug. 6·242	2	Aug. 7·952	
		N.G.C. 7027	0 59 E.	Aug. 6·308		Aug. 7·892	
		N.G.C. 7662	1 31 E.	Aug. 6·382			
		I.C. 4997	3 15 W.	Aug. 6·451			
W543*	70	N.G.C. 7027	1 18 W.	Aug. 11·389	4	Aug. 12·779	
		N.G.C. 7662	0 23 W.	Aug. 11·446		Aug. 12·831	
W567*	80	N.G.C. 7662	3 20 E.	Aug. 29·243	4	Sept. 4·839	Sept. 3·934
		N.G.C. 6572	3 33 W.	Aug. 29·311		Sept. 4·899	Sept. 4·747
W568*	80	N.G.C. 7662	0 02 W.	Aug. 29·383	4	Sept. 5·026	Sept. 3·809
		N.G.C. 7027	4 06 W.	Aug. 29·456		Sept. 4·965	Sept. 3·869
W569	80	N.G.C. 7662	4 07 E.	Aug. 30·208	0	Sept. 5·273	Sept. 1·960
						Sept. 5·212	Sept. 2·018
W570	80	N.G.C. 7662	2 42 E.	Aug. 30·266	0	Sept. 5·089	Sept. 2·769
		N.G.C. 6572	4 05 W.	Aug. 30·331		Sept. 5·150	Sept. 2·829
W571	80	N.G.C. 7662	0 46 W.	Aug. 30·411	0	Sept. 6·231	Sept. 1·836
						Sept. 6·169	Sept. 1·896
W576*	80	I.C. 4593	3 42 W.	Sept. 2·223	4	Sept. 5·978	
		N.G.C. 6572	3 11 W.	Sept. 2·285		Sept. 6·074	
		N.G.C. 7662	0 23 E.	Sept. 2·355			
		I.C. 4997	4 18 W.	Sept. 2·422			
		N.G.C. 7027	5 04 W.	Sept. 2·486			

is the N.G.C. number of the nebula, the second is the hour angle of the nebula at the middle of the exposure, the third is the time (U.T.) in days and fractions of a day of the middle of the exposure, and the fourth is the definition of the nebular image on a scale of 0 to 5. The fourth main column gives the time (U.T.) of the middle of the calibration exposure, the first entry referring to the blue spectrum, the second to the ultra-violet

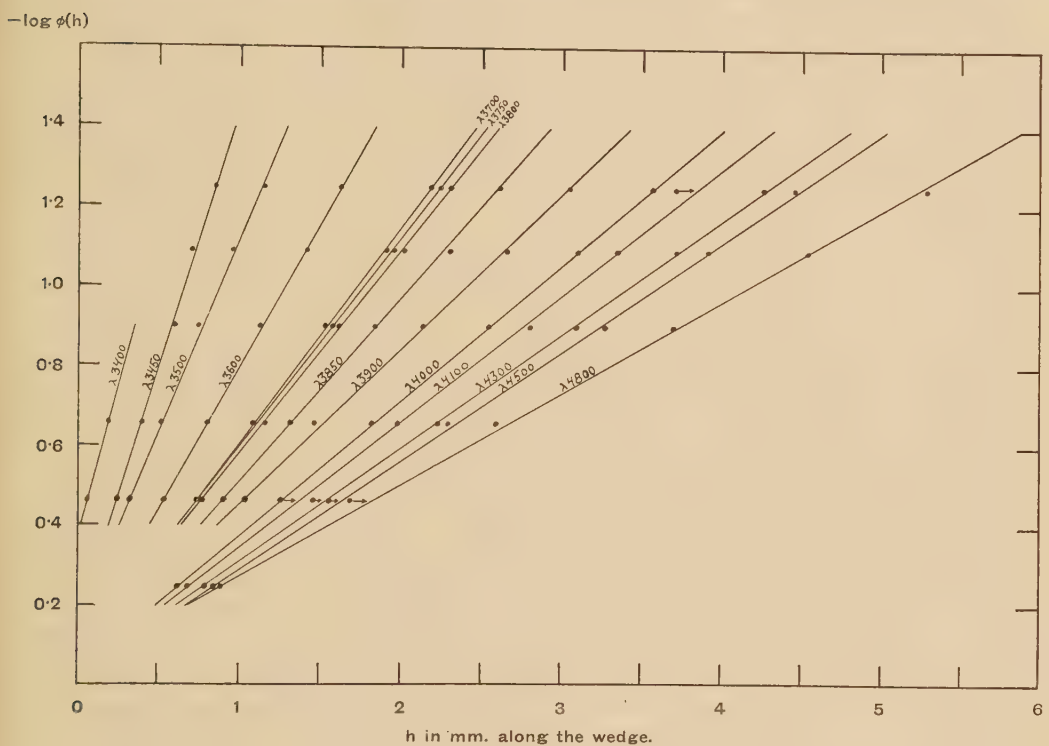


FIG. 1.—*Wedge calibration.* ϕ(h) *is the transmission of the optical wedge at a distance* h *from the thin edge of the wedge.*

spectrum. Similarly the last column gives the time of the blue and ultra-violet standardization exposures.

"This observational investigation was made possible by a grant from the Milton Fund of Harvard, and by some financial assistance from the Dominion Government. In addition, acknowledgment is due to Dr. J. S. Plaskett for giving facilities for the use of the 72-inch telescope, and to Dr. L. S. Ornstein of Utrecht for providing the standard lamp of known energy distribution."

(*b*) *Calibration of the Wedge.*—The wedge calibration plates, described above, were measured for wedge constant in the ultra-violet region λ 3400 to λ 3900 by the method described elsewhere.* Since the observed points [log $\phi(h)$ vs. h] in fig. 1 may best be represented as lying on straight lines, the transmission, $\phi(h)$, of this wedge is accurately expressed as $10^{-h\sigma_\lambda}$, where

* H. H. Plaskett, *M.N.*, **91**, 882, 1930.

h is the distance from the thin edge of the wedge, and σ_λ is a constant at any one wave-length.* A least squares solution for the slope of the lines gives the values and errors of σ_λ in Table II.

TABLE II

Wedge Constant

λ	3400	3450	3500	3600	3700	3750	3800
σ_λ	1·570	1·332	0·965	0·7164	0·5374	0·5224	0·5048
error	...	± ·034	± ·021	± ·0060	± ·0052	± ·0068	± ·0040
% error	...	2·6%	2·2%	0·8%	1·0%	1·3%	0·8%
λ	3850	3900	4000	4100	4300	4500	4800
σ_λ	0·4564	0·3840	0·3394	0·3198	0·2844	0·2714	0·2266
error	± ·0040	± ·0046	± ·0006	± ·0068	± ·0026	± ·0028	± ·0040
% error	0·9%	1·2%	0·2%	2·1%	0·9%	1·0%	1·8%

One difficulty encountered in the calibration was due to the exceptionally curved image of the slit on the plate (due to the short focal length of the collimator in the spectroscope). When the microphotometer † was run across the spectrum at a certain wave-length, λ (λ having been measured at the *edge* of the spectrum), it was actually measuring plate density due to a different wave-length, $\lambda - \Delta\lambda$, in the centre of the run. The effect is to shift central points on the log $\phi(h)$ curves toward curves corresponding to more violet light, but since the slit has a width of 40 A., while the curvature amounts to only $\Delta\lambda = 15$ A. (at λ 4000), the error introduced in the log $\phi(h)$ curves is small.

The pronounced hump in the values for σ_λ at λ 3800 was carefully checked by determinations on either side at λ 3750 and λ 3850, and found to be undeniably present; the absorption coefficient of the "neutral tint" glass does not therefore increase smoothly with decreasing λ.

(*c*) *Characteristic Curves.*—The characteristic curves (plate transmission vs. log intensity) were derived separately for each plate from microphotometer tracings across wedge spectra. They were found to vary considerably in form and slope with the wave-length, especially in the ultra-violet region. Over the range λ 3450 to λ 5000, the slope of the linear part changed fairly smoothly from 0·80 to 1·23, being 1·00 at λ 3900. Finally the characteristic curves were plotted for the wave-lengths: λ 3450, λ 3500, λ 3600, λ 3700, λ 3900, λ 4300, λ 4700 and λ 5000 for each plate. Special care was taken over the exact measurement of the "toes" of the curves which were used in some parts of this investigation. In the succeeding reduction, intensity

* R. O. Redman, *M.N.*, **95**, 744, 1935, finds a "wedge constant" varying with h.

† For a description of the instrument used, see H. H. Plaskett, *M.N.*, **95**, 160, 1934.

ratios for any wave-length, λ, were interpolated between values from two characteristic curves corresponding to wave-lengths on either side of λ.

There is a correction * to apply to these curves (as obtained from wedge spectra) due to the possibly uneven illumination along the wedge (*i.e.* along the slit). This uneven illumination was taken quantitatively as being that observed in wedge-less spectra used in the wedge calibration (see section (*b*) above), since the plates were exposed with identical instrumental conditions. The correction was found to be greater than 2 per cent. only for the λ 3900 curve, which was changed accordingly for all four plates, and in the other curves it was neglected, causing an error of less than 2 per cent. in the desired values of log I.

The error in the ratio, $(I/\mathcal{J})$, of two intensities due to the inaccuracies in characteristic curve is given by

$$\Delta\left(\frac{I}{\mathcal{J}}\right) = 2{\cdot}3 \log\left(\frac{I}{\mathcal{J}}\right)\Delta \log\left(\frac{I}{\mathcal{J}}\right) = 2{\cdot}3 \log\left(\frac{I}{\mathcal{J}}\right)[\Delta^2\sigma + (2 \text{ per cent.})^2]^{\frac{1}{2}}, \qquad (1)$$

where Δ's refer to percentage errors, and the two errors, $\Delta\sigma$ and the 2 per cent. due to non-uniform illumination, have been combined in the usual manner.

(*d*) *Standardization.*—The nebular plates used were standardized by the spectrum of a standard lamp on two of them (W567 and W568). It was necessary to standardize the other two by a central strip 3″·3 of arc wide in the spectrum of N.G.C. 7662 which appeared, well exposed, on all four plates. That is, the intensity distribution of the nucleus (plus superposed nebula) of N.G.C. 7662 was accurately determined from two plates having a laboratory spectrum as standard, and this star spectrum then used to standardize the other two plates. The reduction from microphotometer tracing to spectral intensity is, in either case, a comparison of a blackening on the plate produced by an unknown with one produced by a known intensity (in the same exposure time) using the characteristic curves to obtain intensity ratios.

In the case of plates W567 and W568, the intensity distribution of the standard lamp is calculated by Planck's formula, $\mathcal{J} \propto \lambda^{-5}/(e^{c_2/\lambda T} - 1)$, from a colour temperature, T, derived from an empirical curve (of T as a function of current) supplied by the Utrecht Physical Laboratories. The currents used, 4 and 5 amperes, correspond to temperatures of 2180° K and 2540° K respectively, the higher one having been used to blacken the plate further to the ultra-violet in the same exposure time.

The usual error in brightness temperatures as determined by standardizing laboratories is ± 20° C. If we assume the error in these colour temperatures to be of the same order (*i.e.* 1 per cent.), the percentage error in the ratio of intensities at $T = 2300°$ K near $\lambda_0 = 4000$ A. is

$$\Delta\frac{\mathcal{J}(\lambda_0)}{\mathcal{J}(\lambda_0+\lambda)} = \frac{1{\cdot}432}{T}\left[\frac{\lambda}{\lambda_0(\lambda+\lambda_0)}\right]\Delta T$$

$$= 3{\cdot}9 \text{ per cent. in a range of } 1000 \text{ A.}$$

(*i.e.* $\lambda = 1000$). This error differs from that of the characteristic curve in

* H. H. Plaskett, *M.N.*, **96**, 402, 1936.

being a function of the wave-length (approximately a linear function), as are also the systematic errors computed in the two following sections.

In the case of plates W543 and W576, $\mathcal{J}$, the standard intensity distribution, was first determined from two sets of microphotometer tracings along (referred to as "A") and across ("cross-section") the centre of the N.G.C. 7662 spectrum on plates W567 and W568. The corrections of sections (*e*) and (*f*) were applied, the difference in atmospheric absorption due to different zenith distance being taken into account. Results are given in Table III for the wave-lengths we are here concerned with.

TABLE III

Values of $I\lambda$ along the Centre Strip of the N.G.C. 7662 Spectrum used for Standardization

	λ 4800	λ 4416	λ 4270	λ 4152	λ 4062	λ 3927	λ 3680	λ 3650	λ 3630	λ 3580	λ 3530	λ 3480
W567 Cross-section	1·18	1·25	1·17	1·12	1·18	1·28	1·77	...	2·13	2·02	1·94	2·00
W567 A.	1·18	1·21	1·21	1·12	1·22	1·31	1·77	2·06	2·36	2·12	2·08	1·27
W568 Cross-section	1·22	1·33	1·17	1·32	1·33	1·29	1·51	...	1·95	1·69	2·02	2·18
W568 A.	1·23	1·30	1·28	1·36	1·34	1·25	1·62	2·08	2·08	1·87	2·16	2·07
Mean	1·20	1·27	1·21	1·23	1·27	1·28	1·67	2·07	2·13	1·93	2·05	1·88
Root mean error	1·3%	2·4%	2·6%	6·3%	3·5%	1·2%	4·7%	...	4·9%	6·0%	2·8%	13·6%
Limit to systematic error	12·4%	8·6%	3·5%	2·0%	1·0%	0·0%	1·6%	1·8%	2·0%	2·3%	2·4%	2·7%

The emission-line intensities in N.G.C. 7027 and I.C. 4997 on plates W543 and W576 were measured only relative to lines of N.G.C. 7662, which had in turn been accurately determined (see Table VII, *Part II*) relative to the laboratory standard lamp from plates W567 and W568. The central star spectrum was too faint to use as standard for the lines.

(*e*) *Reflectivity of the Telescope Mirror.*—The reflectivity ratio of mirror to $MgCO_3$ block was determined from the diffuse skylight spectra described in section (*a*), by taking microphotometer tracings along the dispersion of each such spectrum and across wedge spectra for calibration at seven wave-lengths. Intensity ratios read from the characteristic curves of the plates gave values of the reflectivity, $R = I_{\text{mir}}/I_{MgCO_3}$, as a function of wave-length at the two dates July 28 and August 31. These are plotted in fig. 2, and indicate, surprisingly, a possible absorption band in the $MgCO_3$ at λ 3700. Values of R used were interpolated for the dates on which the nebular plates were exposed, assuming a linear tarnishing of the mirror.

The rather large accidental error in R resulting from the above determination amounts to 7·6 per cent. for the range of 1500 A.; that is, with the relative values set at one end of the range, the deviations of different determinations from the mean at the other end lead to the error of 7·6 per cent. in intensity. It will be assumed that the limiting systematic

error due to this inaccuracy in R is a linear function of wave-length amounting to

$$\Delta R = 5{\cdot}1 \text{ per cent. per } 1000 \text{ A.}$$

(*f*) *Atmospheric Extinction.*—The transmission of the atmosphere, a_λ, was computed from Fowle's Tables,* using the average barometric pressure,

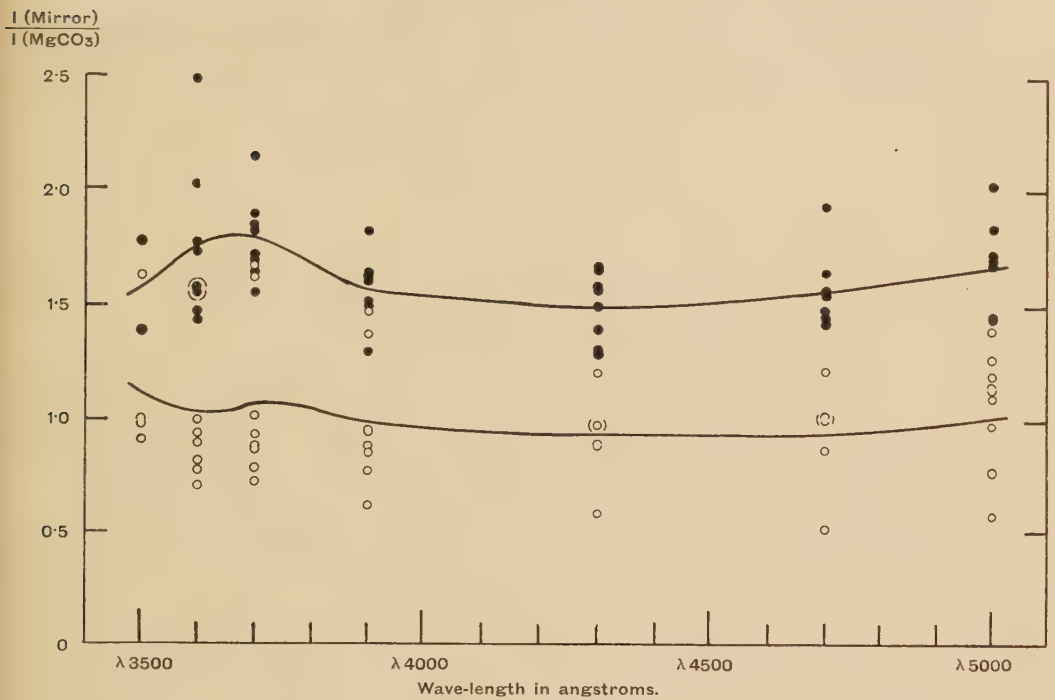


FIG. 2.—*Reflectivity. The ratio of the intensity of skylight reflected from the 72-inch telescope mirror to the intensity reflected from the $MgCO_3$ block is plotted against wave-length for two dates. The upper curve and filled circles are for July 28, the lower curve and open circles for August 31. There are a number of open circles among the filled ones between λ 3500 and λ 4000 causing the high position of the August 31 curve at those wave-lengths.*

$B = 75$ cm., and water content of the air, $w = 1{\cdot}26$ cm., for Victoria during the summer months. The relative transmission is then $a_\lambda^{-\sec z}$ where z is the zenith distance. Fowle's table shows no serious deviations (above λ 3400) from more recent observations by Pettit.† An estimate of the error may be taken as the difference between the calculated a_λ and observations made by Plaskett ‡ in Victoria at the same time of year in 1921. From his Table 13 the error amounts to 3·6 per cent. in 1000 A., again assumed a linear function of wave-length.

(*g*) *Combination of Corrections to Measured Intensities.*—Directly from the microphotometer tracings of nebular and standard spectrum on any plate with the help of the characteristic curves for that plate, we measure

* F. E. Fowle, *Ap. J.*, **38**, 396, 1913.
† E. Pettit, *Ap. J.*, **75**, 185, 1932.
‡ H. H. Plaskett, *Pub. D.A.O.* (Victoria), **2**, 249, 1923.

the ratio of apparent intensities, $I'/\mathcal{J}'$ (nebula to standard). Combining the information in sections (*d*), (*e*) and (*f*) gives the intensity ratio in terms of the ratio of true nebular to true standard intensity as $I'/\mathcal{J}' = Ra^t(I/\mathcal{J})$ or

$$I = \left(\frac{\mathcal{J}}{Ra^t}\right)\left(\frac{I'}{\mathcal{J}'}\right) \tag{2}$$

for continuous spectrum intensities. For nebular line spectrum compared to continuous standard spectrum there must be included an additional factor due to the non-linear dispersion of a prism spectrograph, so that

$$I = \left(\frac{\mathcal{J}}{Ra^t}\right)\left(\frac{I'}{\mathcal{J}'}\right)\frac{(\lambda - \lambda_0)^2}{C} \tag{3}$$

for line intensities, where

I = true nebular intensity.
$\mathcal{J}$ = known standard intensity.
R = reflectivity ratio, mirror to $MgCO_3$.
a = atmospheric transmission.
t = sec $z \propto$ air mass.

C, λ_0 are constants in the Hartmann formula for dispersion, $n_0 - n = C/(\lambda - \lambda_0)$, n being measured in mm. on the plate. These constants were accurately determined from an iron arc spectrum taken with the same spectrograph.

Combination of all the systematic errors in the usual manner leads to

$$\Delta^2 I = \Delta^2\mathcal{J} + \Delta^2 R + t^2\Delta^2 a + 2{\cdot}3(\Delta^2\sigma + 4)\log\left(\frac{I'}{\mathcal{J}'}\right), \tag{4}$$

where the Δ's refer to percentage errors, and the term $\Delta[(\lambda - \lambda_0)^2/C]$ has been neglected as it is less than 1 per cent. The first three terms on the right of (4) vary linearly with the wave-length. The last varies with the intensity ratio measured, and is appreciable only in the ultra-violet where σ_λ is less accurately determined.

Part II—Continuous and Line Intensities in the Nebulæ

(*a*) *Cross-sections of N.G.C. 7662.**—Microphotometer tracings were taken across the spectrum of N.G.C. 7662 perpendicular to the dispersion between points accurately marked on either side of the spectrum (by a needle on a hinged arm attached to a Hilger micrometer). The wave-lengths listed in Table IV, corresponding to these points, had been carefully chosen (from a spectrum map of this nebula on which all of Wright's † lines with the known slit width were plotted) as the only points in the spectrum free from line emission.

The tracings were then reduced to intensities by the methods outlined

* For a photograph of the nebula, see H. D. Curtiss, *Pub. Lick Obs.*, **13**, 57, 1913, plate xxiv.

† W. H. Wright, *loc. cit.*

in *Part I*, a typical cross-section being reproduced as a broken line in fig. 3. Between ten and thirty points were reduced on each such cross-sectional curve. The nuclear star spectrum superposed on the nebular spectrum (and used for standardization) is represented by the peak at the centre. Abscissæ are measured in seconds of arc from the nucleus, positive to the west (right) and negative to the east (left). The area under the curve represents total energy in the light of wave-length λ 3630 from about half the nebula (since the wide slit integrates over half the nebular image).

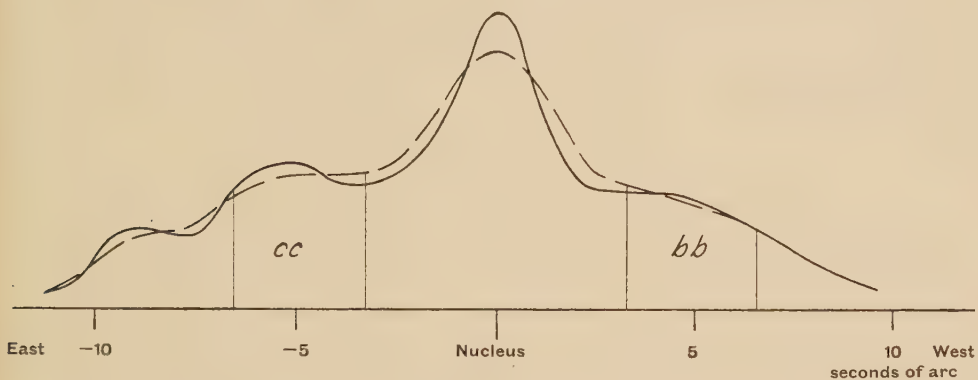


FIG. 3.—*Cross-section of the continuous spectrum of N.G.C. 7662 at λ 3630 from Plate W568. Measured intensity (the broken line) of λ 3630 light is plotted against distance along the slit from the nuclear star* (i.e. *east to west on the nebular disc*). *The full line represents the intensity values corrected for unsteadiness of the image as described in Part II* (*c*). *The areas "bb" and "cc," which represent intensities between 3″·3 and 6″·6, and −6″·6 and −3″·3, are indicated.*

(*b*) *Intensity Distribution in Wave-length : N.G.C. 7662.*—To obtain curves of intensity distribution as a function of wave-length in two parts of N.G.C. 7662, areas were measured with a planimeter under the cross-section intensity curves between $x = +3''{\cdot}3$ and $6''{\cdot}6$ (referred to as "*bb*" for convenience) and between $x = -6''{\cdot}6$ and $-3''{\cdot}3$ ("*cc*"). These parts of the nebula were chosen as it was evident that the east and west sides had somewhat different spectra. The distance of the strip 3″·3 to 6″·6 from the nucleus is such as to include as little as possible of the blurred nuclear spectrum, and yet to record the maximum nebular intensity.

An independent reduction was made from microphotometer tracings taken along the spectrum parallel to dispersion at points 4″·9 of arc from the nucleus on either side (accurately marked in the Hilger micrometer). The length of the microphotometer slit was set to cover as exactly as possible 3″·3, *i.e.* the strips between +3″·3 and +6″·6 (*bb*) and −6″·6 and −3″·3 (*cc*). The intensity results agreed within 2 or 3 per cent. with the "cross-section" values of *bb* and *cc* above. Values of *bb* and *cc* from the two different reductions were averaged for each plate, and are given in Tables IV and V for twelve wave-lengths from four plates. These have been corrected for unsteadiness of the image, as described in the next section. Their combination to a mean value is discussed in section (*e*).

(*c*) *Correction for Unsteadiness of the Image.*—A correction is now made

to the cross-section curves of N.G.C. 7662 for movement of the image along x due to atmospheric unsteadiness and guiding error during the long exposure time of 80 minutes. The blurring effect of this movement may be represented by a scattering function $\phi(x)$, such that the observed curve, $I'(x)$, is given by

$$I'(x) = \int_{-\infty}^{+\infty} I(\xi)\phi(x-\xi)d\xi \qquad (5)$$

(where $\phi(x) = (b/\sqrt{\pi})e^{-b^2x^2}$, and $I(x)$ is the true intensity curve) as has been used by Wanders * in a slightly different form for observations on sunspots. The second approximation to the solution of (5) for $I(x)$ is given by Burger and Van Cittert † as

$$I(x) = 2I'(x) - \frac{b}{\sqrt{\pi}}\int_{-\infty}^{+\infty} I'(\xi)e^{-b^2(x-\xi)^2}d\xi, \qquad (6)$$

where the integral can be determined graphically when b, the scattering parameter, is known. b is determined here from the spectrum of the nuclear star, which, if undisturbed, would be a narrow peak, its width being due to diffraction in the instruments. The integrated unsteady cross-section of the star will be observed as approximately a pure Gaussian curve, $\phi(x)$, multiplied by the original intensity of the star image and superposed on the nebular intensity. To obtain b it is necessary to subtract the mean nebular background and measure the width of the star peak at several heights. From $I_s\phi(x) = I_s(b/\sqrt{\pi})e^{-b^2x^2}$, b is found to be $b = \sqrt{-4\ln q/(2x)^2}$, where q is the fractional height at which $2x$, the width of the star peak, is measured. Values of b were averaged from all cross-sections on each plate, resulting in the following mean values for plates W543, W567, W568 and W576, respectively: 0·690 ± ·023, 0·538 ± 0·047, 0·570 ± ·045 and 0·536 ± ·014, these values having the physical dimensions of [seconds of arc]$^{-1}$. The root mean square errors, listed above, average about 5·6 per cent. No systematic change of b with wave-length could be found. b is a measure of the definition of the plate: the greater b, the better the definition, as was evident to the eye upon comparing the definitions of W543 ($b = 0{\cdot}69$) and W567 ($b = 0{\cdot}54$).

The full line of fig. 3 is the corrected cross-section intensity curve from equation (6). Areas under the full line between 3″·3 and 6″·6, −6″·6 and −3″·3 are the corrected values of *bb* and *cc*. The effect of this correction was in most cases less than 10 per cent. because *bb* and *cc* were chosen far from the nucleus. The separated total nebular intensity is used, however, in *Part IV*, where this correction is of importance. Corrected *bb* and *cc* values from four plates are listed in Tables IV and V, with mean values and root mean errors.

(*d*) *Continuous Spectrum in the Planetary Nebulæ, N.G.C. 7027, N.G.C. 6572, I.C. 4593 and I.C. 4997.*—Cross-sections of these nebulæ proved useless: N.G.C. 7027 is very irregular, N.G.C. 6572 and I.C. 4593 are

* A. J. M. Wanders, *Zs. f. Astrophys.*, **8**, 108, 1934.

† Burger and Van Cittert, *Zs. f. Phys.*, **79**, 722, 1932.

TABLE IV

Intensities per Angstrom of Continuous Spectrum 4″·9 West of the Nucleus in N.G.C. 7662 (bb)

Plate	Weight	λ 4800	λ 4416	λ 4270	λ 4152	λ 4062	λ 3927	λ 3680	λ 3650	λ 3630	λ 3580	λ 3530	λ 3480
W543	0·75	0·66	0·64	0·58	0·58	0·59	0·60	0·57	0·82	0·86	0·73	0·68	0·34
W567	1·00	·77	·41	·59	·49	·61	·75	(·45)	(·93)	(·75)	(·64)	(·66)	(·55)
W568	1·00	·66	·44	·51	·48	·54	·40	·72	·83	·90	·75	·73	·71
W576	0·75	·65	·48	·45	·53	·51	·49	·61	·70	·85	·80	·86	·69
Mean I_λ		0·695	0·483	0·535	0·517	0·566	0·563	0·613	0·816	0·856	0·740	0·740	0·590
I_λ rel. to λ 3927		1·23	0·86	0·95	0·92	1·01	1·00	1·09	1·45	1·52	1·31	1·31	1·05
Corr. for slit width		1·23	0·86	0·95	0·92	1·01	1·00	1·04	1·48	1·52	1·31	1·31	1·05
Acc. error		7·2%	14·5%	9·4%	7·7%	7·1%	16·0%	11·4%	6·1%	4·7%	5·6%	8·1%	23·8%
Sys. error		12·4%	8·6%	3·5%	2·0%	1·0%	0·0%	1·6%	1·8%	2·0%	2·3%	2·4%	8·7%

See Section II (*h*) for continuous intensities relative to line intensities.

(Values in brackets in the ultra-violet are given 0·5 weight because of under-exposure.)

TABLE V

Intensities per Angstrom of Continuous Spectrum 4″·9 East of the Nucleus in N.G.C. 7662 (cc)

Plate	Weight	λ 4800	λ 4416	λ 4270	λ 4152	λ 4062	λ 3927	λ 3680	λ 3650	λ 3630	λ 3580	λ 3530	λ 3480
W543	0·75	0·81	0·78	0·66	0·64	0·69	0·68	0·79	1·26	1·28	1·07	0·97	0·56
W567	1·00	·90	·71	·70	·52	·61	·76	(·63)	(1·47)	(1·57)	(1·21)	(0·57)	(0·50)
W568	1·00	·80	·60	·64	·64	·70	·43	·96	1·24	1·18	1·11	1·11	0·78
W576	0·75	·67	·59	·56	·59	·62	·54	·77	1·26	1·41	1·15	1·16	0·85
Mean I_λ		0·803	0·669	0·646	0·595	0·654	0·600	0·813	1·29	1·33	1·13	1·00	0·696
I_λ rel. to λ 3927		1·34	1·11	1·08	0·99	1·09	1·00	1·36	2·15	2·22	1·88	1·67	1·16
Corr. for slit width		1·34	1·11	1·08	0·99	1·09	1·00	1·26	2·26	2·22	1·88	1·67	1·16
Acc. error		8·7%	10·4%	6·2%	8·4%	6·1%	21·7%	12·3%	4·6%	9·0%	3·5%	16·0%	18·7%
Sys. error		12·4%	8·6%	3·5%	2·0%	1·0%	0·0%	1·6%	1·8%	2·0%	2·3%	2·4%	2·7%

(Values in brackets in the ultra-violet are given 0·5 weight because of under-exposure.)

centrally condensed, no distinct boundary between star and nebula being evident, and I.C. 4997 is a stellar nebula.* Therefore microphotometer tracings were simply taken parallel to dispersion along the centre of the spectra, and included any nuclear star spectrum present. The nuclear star of N.G.C. 7027 (perhaps obscured by dark occulting matter responsible for the dark lanes across the nebula) is never recorded in direct photographs of the nebula, and it is therefore safe to assume that these results represent pure nebular spectrum. The only spectrum of I.C. 4997 is too under-exposed to give reliable intensities in the continuous spectrum.

Intensities, accurately reduced by the methods outlined in *Part I*, are listed in Table VI.

* For photographs of these nebulæ, see H. D. Curtiss, *loc. cit.*, plates xxiii, xxii, xvi and xiii.

TABLE VI

Intensities per Angstrom of Continuous Spectra in N.G.C. 7027, N.G.C. 6572, I.C. 4593

N.G.C. 7027 Plate	Weight	λ 4800	λ 4416	λ 4270	λ 4152	λ 3927	λ 3680	λ 3650	λ 3630	λ 3580	λ 3530	λ 3480
W543	0·75	1·13	0·81	0·74	0·63	0·55	0·74	0·97	0·95	0·66	0·56	0·15
W568	1·00	1·21	·67	·67	·54	·56	·69	1·15	1·00	·71	·28	·41
W576	0·75	1·15	·69	·71	·65	·66	·64	0·86	0·79	·60	·68	·34
Mean I_λ		1·17	0·720	0·702	0·600	0·586	0·690	1·005	0·923	0·663	0·482	0·343
I_λ rel. to λ 3927		2·00	1·23	1·20	1·03	1·00	1·18	1·72	1·58	1·13	0·82	0·59
Corr. for slit width		2·00	1·23	1·20	1·03	1·00	1·12	1·78	1·58	1·13	0·82	0·59
Acc. error		3·2%	9·2%	3·9%	9·4%	9·6%	5·5%	13·2%	10·2%	7·1%	37%	41%
Sys. error		12·7%	8·7%	3·4%	1·2%	0·0%	2·0%	2·2%	2·3%	2·7%	2·9%	3·2%
N.G.C. 6572												
W567	1·00	3·11	3·08	3·08	2·81	2·90	5·53	7·05	7·43	6·43	5·83	5·24
W576	0·75	3·09	3·57	3·62	3·42	2·57	4·94	6·92	6·68	6·23	6·20	5·20
Mean I_λ		3·10	3·29	3·32	3·07	2·76	5·27	7·00	7·12	6·35	5·99	5·22
I_λ rel. to λ 3927		1·12	1·19	1·20	1·11	1·00	1·91	2·54	2·58	2·30	2·17	1·89
Corr. for slit width		1·12	1·19	1·20	1·11	1·00	1·81	2·62	2·58	2·30	2·17	1·89
Acc. error		0·3%	7·3%	8·1%	9·8%	5·8%	6·3%	1·0%	5·2%	1·6%	3·0%	0·4%
Sys. error		12·9%	8·8%	3·4%	1·2%	0·0%	2·0%	2·2%	2·3%	2·7%	2·9%	3·1%
I.C. 4593												
W576		...	...	...	(1·00)	(1·00)	1·12	1·20	1·25	1·24	1·11	1·15
Corr. for slit width		...	...	...	(1·00)	(1·00)	1·08	1·24	1·25	1·24	1·11	1·15

(*e*) *Combination of Observations: Correction for the Spectroscope Slit Width.*—The weights used in combining the results from four plates (noted in Tables IV, V and VI) are 1·00 for plates W567 and W568, which were standardized directly by the laboratory lamp, and 0·75 for the plates W543 and W576, which were standardized indirectly through the star spectrum of N.G.C. 7662 and which were well exposed for the continuous spectrum. In the ultra-violet part of the N.G.C. 7662 spectrum on plate W567, a weight of 0·5 was introduced because of the under-exposure of the nebular spectrum there.

The accidental errors are derived by least squares methods from the deviations of observations from the mean. Systematic errors are computed by equation (4).

A small correction must be made to the final intensity distribution owing to the wide slit. Where the continuous spectrum changes slope rapidly at the head of the Balmer series, the averaging effect of the 40-A-slit will be appreciable. The correction,* determined from an integral equation similar to equation (5) above, amounts to about −3·5 per cent. at λ 3680

* F. L. O. Wadsworth, *Ap. J.*, **3**, 321, 1896.

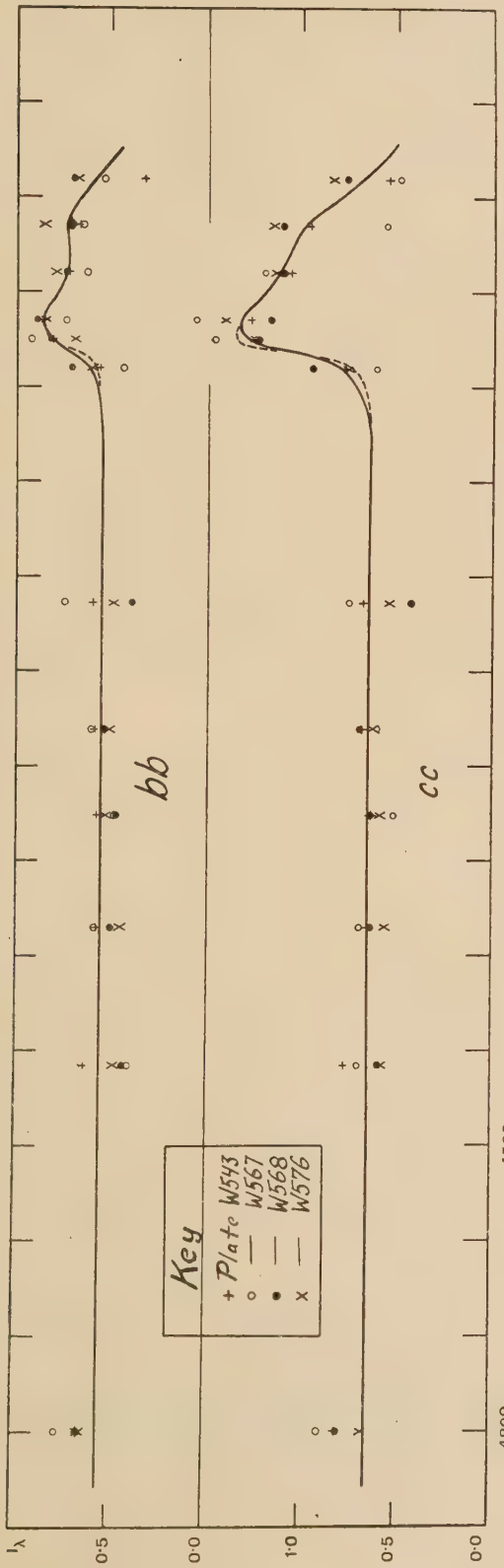


FIG. 4.—*Observed intensities in the continuous spectrum of N.G.C. 7662. Values from four different plates are plotted against wave-length, the upper curve representing intensities on the west side, and the lower those on the east side, of the nebula. From λ 4800 to λ 3900 the full line represents a constant intensity, the mean of observations at all wave-lengths. In the ultra-violet the full line is drawn through the mean at each wave-length and the broken line illustrates the correction for spectroscopic slit width.*

and +5 per cent. at λ 3650. It has been applied to obtain the final values of I_λ labelled in Tables IV, V and VI "corrected for slit width." Its effect can best be seen in fig. 4, where Tables IV and V are plotted.

(*f*) *Line Cross-sectional Forms and Total Intensities.*—With such a wide slit (0·5 mm.) and moderate dispersion (about 80 A. per mm. near

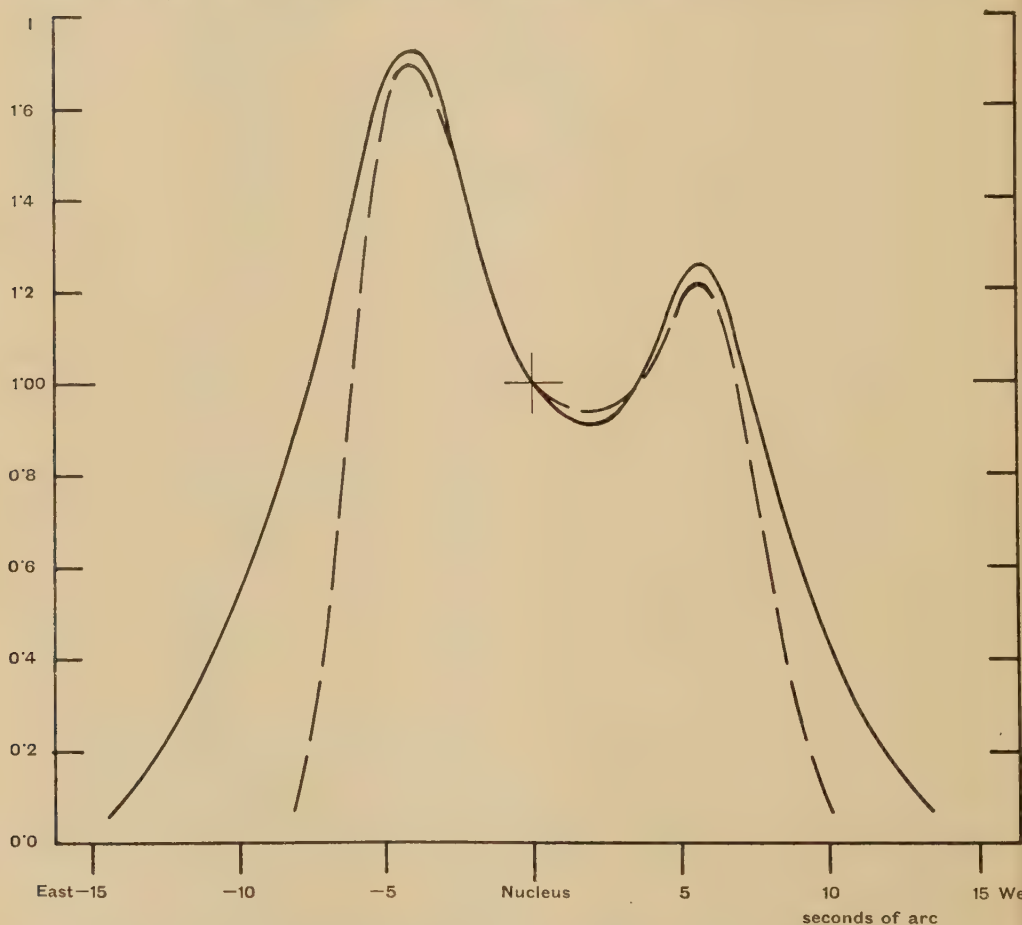


FIG. 5.—*Cross-sections of pure emission-line images in the spectrum of N.G.C. 7662. The continuous curve is the mean cross-section of the Balmer lines, and the dashed curve that of λ 4686, both having been corrected for overlapping intensity (therefore no nuclear star peak is present) and for unsteadiness of the image (as described in Part II (c)). Intensities are plotted relative to the pure nebular intensity at the centre of the nebular disc.*

λ 3900) as were used for these spectra, emission lines overlap in many cases, and blackening of the plate due to continuous spectrum is superposed on every line as well. However, accurate determinations of pure line intensities have been made in the following manner :—

For N.G.C. 7662, cross-sections of the spectrum at the centre of, and just beside, each line image have been reduced by methods described in *Part I.* Then the overlapping intensity given by the cross-section just beside the

line image was subtracted graphically, point for point, leaving the true cross-sectional curve and total intensity (the remaining area). Total line intensities from plates W567 and W568 are averaged and entered in column 5 of Table VII. Percentage errors, given in column 6, are derived from the difference of the results from two plates, and give only an indication of the accuracy. The intensity of H_β is poor because of over-exposure. The intensities not corrected for overlap are given for comparison in column 4.

The line cross-sectional curves from four plates were corrected for unsteadiness of the image by the method of *Part II* (*e*) and an average curve drawn. Fig. 5 illustrates accurately a mean hydrogen line cross-section (the full line) with that of the *He* II λ 4686 line (the broken line) for comparison. There was no difference in form between the *H* lines and the Ne^{++} lines, λ 3969 and λ 3869. λ 3445 had a poorly determined cross-section but was markedly similar to λ 4686.* The curves in fig. 5 represent east-west intensity sections at the centre of the pure N.G.C. 7662 monochromatic images, and are of greater accuracy than Berman's contour maps,† which include the nuclear star peak, subtracted here with the overlapping intensity of each line. The curves should prove useful for investigations concerning the structure and equilibrium of planetary nebulæ. The error is less than 10 per cent. (calculated from mean accidental deviations of four plates combined with the small systematic error due to characteristic curve).

For N.G.C. 7027 and I.C. 4997, contours of overlapping lines were reduced from microphotometer tracings along the spectrum parallel to dispersion. The contour (as distinct from the cross-section) of a pure line could then be accurately fitted to each of the lines in a blend, and the intensities separated. In this manner the values in columns 7 and 9 of Table VII have been derived. Results from three plates were combined with the weights discussed in *Part II* (*e*) for the N.G.C. 7027 intensities listed. Line intensities of I.C. 4997 depend on one plate only.

(*g*) *Comparison with Other Determinations.*—Table VIII offers in compact form the comparison of these line intensities with those measured by Plaskett ‡ and Berman § for the same nebulæ. In the visual region the agreement is everywhere approximately within the experimental error, but in the ultra-violet region of N.G.C. 7027 the present determinations are not in accord with those of Berman. He has used Abbot's energy values for the Sun as standard intensity distribution and an undescribed method of reducing intensities to compare with the spectrum of 16 *Peg.*, and thence with the nebulæ. Apart from the rougher photometric methods used, the main cause of the discrepancy in Berman's line intensity values is the overlapping continuous intensity, which he has neglected to subtract. With wide slit or slitless spectra, the plate blackening produced by even a

* A fact first recognized by Wright, *loc. cit.*

† L. Berman, *Lick Obs. Bull.*, **15**, No. 430, 86, 1930.

‡ H. H. Plaskett, *Pub. D.A.O.* (Victoria), **4**, 121, 1928.

§ L. Berman, *loc. cit.*

TABLE VII

Line Intensities in Planetary Nebulæ

λ	Source *	Classification *	N.G.C. 7662 from 2 Plates			N.G.C. 7027 from 3 Plates		I.C. 4997 from 1 Plate
			Total Measured Intensity	Intensity Corrected for Overlap	Error	Intensity Corrected for Overlap	Error	
4862	H_β	2–4	(3·24)	(2·78)	(10%)	(2·20)	(13%)	(1·76)
4740	*A* IV?	^4S–$^2D_{3/2}$	...	...	...	0·169	19%	
4711	*A* IV?	^4S–$^2D_{5/2}$	0·65	1·54 (0·23 / 1·31)	28% / 7·4%			
4686	*He* II	3–4	2·03			0·955	7·8%	
4641	*N* III	^2D–^2D	...	...	...	0·076	38%	
4363	*O* III	^1D–^1S	0·62	1·32 (0·32 / 1·00)	16% / 1·9%	0·472	24%	1·028
4341	H_γ	2–5	1·61			1·00	18%	1·00
4102	H_δ	2–6	0·71	0·474	5·4%	0·221	6·5%	0·251
4069	*S* II?	^4S–$^2P_{3/2}$	...	...	...	0·046	26%	
3969	*Ne* III $^3P_1$–^1D / H_ϵ 2–7		0·84	0·61	10%	0·328	8%	0·401
3889	H_ζ 2–8 / *He* I ^3S–^3P		0·44	1·91 (0·23 / 1·68)	37% / 7·6%	0·188	11%	0·252
3869	*Ne* III	$^3P_2$–^1D	2·12			1·26	12%	1·125
3836	H_η	2–9	...	...	...	0·073	23%	
3445	*O* III	$^3P_2$–$^3P_2$	0·60	0·44 (0·22 / 0·22)	26% / 31%	0·046	38%	
3426	*Ne* V?	$^3P_2$–^1D	0·37			0·86	21%	
3346	*Ne* V?	$^3P_1$–^1D	...	...	...	0·35	21%	

* I. S. Bowen, *Ap. J.*, **81**, 1, 1935.

TABLE VIII

Nebular Line Intensities: Comparison with Other Determinations *

λ	N.G.C. 7662		N.G.C. 7027		
	Plaskett	T.L.P.	Plaskett	T.L.P.	Berman
4862	2·27 ± ·23	(2·78 ± ·28)	2·6 ± ·3	(2·20 ± ·3)	2·4 ± ·4
4686	1·37 ± ·14	1·31 ± ·10	1·00 ± ·10	0·96 ± ·08	0·91 ± ·15
4363	0·43 ± ·04	0·32 ± ·05	0·58 ± ·06	0·47 ± ·11	0·55 ± ·09
4341	0·89 ± ·09	1·00 ± ·02	0·89 ± ·09	1·00 ± ·18	0·92 ± ·15
4102	0·43 ± ·04	0·47 ± ·03	0·39 ± ·04	0·22 ± ·01	0·51 ± ·08
3969	...	...	...	0·33 ± ·03	0·90 ± ·15
3869	...	...	...	1·26 ± ·15	2·3 ± ·4
3445	...	...	...	0·05 ± ·02	0·61 ± ·10
3426	...	...	...	0·86 ± ·18	2·6 ± ·4
3346	...	...	...	0·35 ± ·07	1·01 ± ·17

* The scales of Plaskett's and Berman's results have been altered so that the sum of intensities of λ 4363 and λ 4341 is the same as in Table VII.

weak continuous intensity affects line intensities by as much as 100 per cent. (*cf.* Table VII). The effect is not avoided by a shorter exposure time revealing only lines on the plate.

(*h*) *Continuous Emission relative to Line Intensities.*—The intensity of nebular continuous relative to line spectrum is not required in the following discussion, but for the sake of completeness we compute the factors necessary to transform intensities in Tables IV, V and VI into values relative to H_γ. As a result of certain simplifying assumptions it may be shown that the measured intensity of continuous spectrum relative to line spectrum is w times too great, where w is the width of the slit. The necessary assumptions are that diffraction may be neglected, that over the 40-A-width of the slit the continuous spectrum intensity is independent of wave-length, and finally that the average value of the surface brightness of the nebula across the slit is a good representation of the surface brightness at any one wave-length (checked by measuring line images in a direction parallel to dispersion). If these assumptions are valid, it follows that, in Tables IV and V,

$$bb = w\int_{3''\cdot3}^{6''\cdot6} I_\lambda(x)dx \qquad cc = w\int_{-6''\cdot6}^{-3''\cdot3} I_\lambda(x)dx,$$

while the line intensities listed in Table VII are given by

$$L = \int_{-\infty}^{+\infty} I_\lambda(x)dx,$$

where $I_\lambda(x)$ is, in both cases, the intensity in light of wave-length λ at a point distant x seconds of arc from the nucleus. By integrating the line intensity cross-sections between the same limits of x as *bb*, we find that *bb* (3927) $= 3{\cdot}29 \times 10^{-3} H_\gamma$. That is, the final values of continuous spectrum intensities per angstrom relative to the λ 3927 intensity given in Table IV must be multiplied by $3{\cdot}29 \times 10^{-3}$ to reduce them to the scale of $H_\gamma = 1$. In Table V the factor is $2{\cdot}50 \times 10^{-3}$, and in Table VI for N.G.C. 7027 it is $0{\cdot}586 \times 10^{-3}$.

Part III—Theoretical Discussion of the Balmer Emission of Hydrogen

(*a*) *Review.*—The consideration of a group of atoms in equilibrium with radiation in an enclosure leads, by "detailed balancing" arguments, to a relation (first derived by Milne *) between the probabilities of absorption (or ionization) and emission (or recapture). This is an atomic relation valid under any circumstances, and it may be applied to atoms in a nebula even though conditions there are very different from those in the ideal enclosure. If, moreover, the free electrons in the nebula resulting from the ionization are assumed to have velocities obeying a Maxwellian distribution law, and if the Kramers-Gaunt absorption coefficient law † ($a_\nu \propto n^{-5}\nu^{-3}$, given by quantum mechanics) is assumed correct, it is possible to predict

* E. A. Milne, *Phil. Mag.*, **47**, 209, 1924.

† H. A. Kramers, *Phil. Mag.*, **46**, 837, 1923 ; Gaunt, *Phil. Trans.*, **229 A**, 200, 1929 ; Menzel and Pekeris, *M.N.*, **96**, 77, 1935.

the continuous emission due to free-bound transitions of hydrogen-like atoms. This theoretical prediction may be expressed as *

$$I_\nu d\nu = n_2 y \frac{Z^4 C_1}{n^3} T_\epsilon^{-3/2} e^{\frac{-h(\nu-\nu_n)}{kT_\epsilon}} d\nu \qquad (7)$$

for free-bound transitions, where C_1 is a known combination of electronic and geometrical constants, n_2 and y are the number of protons and electrons per cm.^{3} respectively, Z the effective nuclear charge, n the quantum number of the final orbit, T_ϵ the free electron temperature in the nebula, and ν_n the frequency of the nth series limit. For the Balmer continuous (hereafter referred to as "Ba_c") intensity distribution of hydrogen, $Z=1$, $n=2$, $\nu_2 = Rc/4 = 8{\cdot}219\ 10^{14}$/sec.

By the same reasoning, relations are derived by Cillié for the emission and absorption connected with free-free transitions. The resulting formula, similar to (7) and based on the same two assumptions of Maxwellian distribution and absorption law, is

$$I_\nu d\nu = n_2 y Z^2 C_2 T_\epsilon^{-\frac{1}{2}} e^{-h\nu/kT_\epsilon} d\nu \qquad (8)$$

for free-free transitions.

(*b*) *Theory and Observation.*—The observational results in Tables IV, V and VI may now be compared with the theoretical equation (7). The visual continuous intensity (referred to as "V_c"), which is very nearly constant throughout the region λ 4800 to λ 3927, and therefore presumably constant down to λ 3480, must first be subtracted from ultra-violet intensities to obtain pure Ba_c. V_c is taken as the mean of I_λ for all wave-lengths from λ 4800 to λ 3927 inclusive, and the observed $I_\lambda - V_c$, *i.e.* the true Ba_c, is shown by full lines in fig. 6 for the two parts of N.G.C. 7662. The broken lines are theoretical values from equation (7) for indicated values of T_ϵ.

The hump in Ba_c intensity at λ 3530 (especially in *bb* of N.G.C. 7662) is within observational error. If it exists, it may be due to a failure of one or other of the two theoretical assumptions. Consider the first of these: a Maxwellian distribution of electron velocities would be preserved by the mechanism of photo-ionization and reçapture on the Lyman orbit alone, since these processes form a unit mechanism. Captures on higher levels and the excitation of the "nebulium" lines, however, disturb the Maxwellian distribution considerably. The number of quanta in "nebulium" lines is found from line intensities divided by $h\nu$, the energy of a quantum, to be twice as great as the number in the Balmer lines, Ba_c and computed Ly_c. [This has been the first opportunity, owing to the measurement of continuous intensity, of computing such a ratio; it would seem to indicate that each free electron is used twice for electron excitation before recapture.] In the extremely rare gas of the nebula, therefore, there might be a very peculiar velocity distribution. A sharp maximum at 2×10^7 cm./sec., for instance (corresponding to λ 3530 in the Ba_c), would produce the observed hump. But laboratory measures of excitation probabilities by

* A formula due to Cillié, *M.N.*, **92**, 824, 1932.

electron impact * indicate a value near 10^{-3}, so that an accurate Maxwellian distribution should be attained by the 1000 or more elastic collisions between each inelastic one.

With regard to the second assumption, an alternative absorption coefficient law of the general form, $a_\nu \propto n^{-5}\nu^{-3+s}(\nu-\nu_0)^{-r}$, † cannot, with any values of s and r, account for a hump of the observed size at λ 3530.

Neither of the assumptions, therefore, seems to be the source of discrepancy. The superposed continuous spectrum assumed to be a constant, V_c, in the range λ 3650 to λ 3480, might, however, vary, especially if V_c

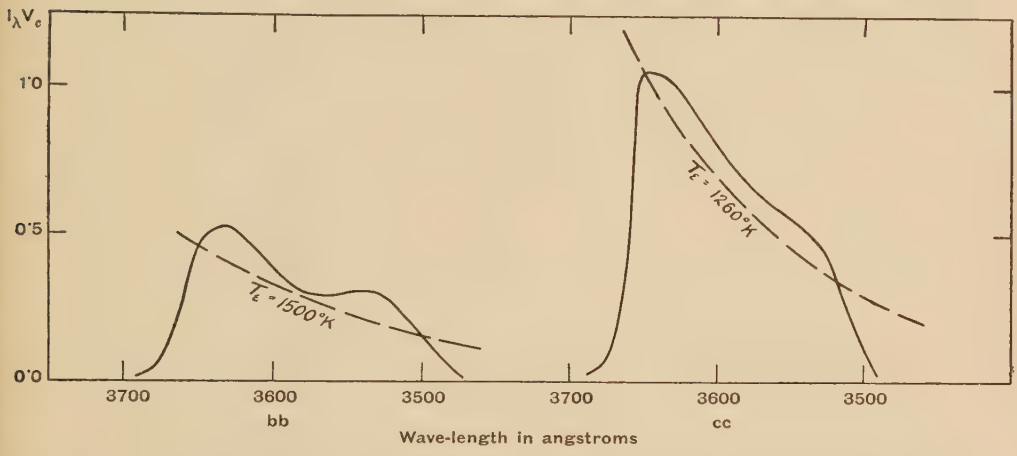


FIG. 6.—*The pure Balmer continuous emission in two parts of N.G.C. 7662. The continuous curve on the left represents the variation of measured intensity minus V_c with wave-length in the western part of the nebula (bb), that on the right represents the variation in the eastern part (cc). The dashed curves are the theoretical predictions for the indicated values of T_ϵ.*

is due to nuclear light. [There is probably Balmer continuous absorption in the star's atmosphere, and if the starlight were scattered in any way so as to give V_c, the dip in intensity would still be present.] When V_c is not subtracted and equation (7) is fitted directly to values of I_λ in the ultra-violet, somewhat higher electron temperatures result. The differences are taken as indications of the accuracy (they do not, however, define upper and lower limits of T_ϵ) ; both sets of values of T_ϵ are listed in Table IX (*a*).

Observation, then, can be said to confirm, within experimental error, theoretical predictions and therefore the two assumptions. The determination of electron temperatures (which is the chief object of this investigation in the ultra-violet) results in surprisingly low figures (near 2000° K) in comparison with the 10,000° K and 20,000° K hitherto supposed.‡

The values of T_ϵ in Table IX (*a*) show, in so far as the data are complete, that higher T_ϵ occurs with lower N_1 and N_2 intensity. A theoretical explanation is clear ; if the excitation of N_1 (λ 5007), N_2 (λ 4959), λ 3969 and λ 3869 is solely due to inelastic impact with free electrons, the emission

* E. G. Dymond, *Proc. Roy. Soc.*, A, **107**, 291, 1924.

† Suggested by the results of Lawrence and Edlefsen, *Phys. Rev.*, **34**, 233, 1929.

‡ H. H. Plaskett, *Pub. D.A.O.* (Victoria), **4**, 121, 1928 ; G. Cillié, *loc. cit.*

of these lines obviously withdraws energy from, or cools, the electron gas. That is, for more O^{++} and Ne^{++} (relative to H), the nebulium lines will be stronger, T_ϵ will be lower, and the H continuous emission will be correspondingly weaker.

TABLE IX (*a*)

*Observed Electron Temperatures and Balmer Line Intensities.**

	N.G.C. 7027	N.G.C. 7662	N.G.C. 6572	I.C. 4593
T_ϵ (deg. abs.)	700 to 1600	1200 to 4000	2700 to 4500	2200 to 11000
H_α	58	...	...	...
H_β	10·0	10·0	10·0	10·0
H_γ	3·7	4·1	5·8	5·6
H_δ	1·2	1·9	3·4	2·6
N_2	42	32	26	14·4
λ 4686	3·9	5·8	0	0
T_s	50000°	40000°	35000°	24000°

* The line intensities in N.G.C. 7027 and N.G.C. 7662 are from H. H. Plaskett (*loc. cit.*) except for H_γ, H_δ and λ 4686, where the values from Table VII have been averaged with Plaskett's, first setting the scale by the sum of λ 4686, λ 4363, H_γ and H_δ. The line intensities in N.G.C. 6572 are all from L. Berman (*loc. cit.*), and in I.C. 4593 from H. H. Plaskett (*loc. cit.*). Nuclear temperatures, T_s, are the most probable values from *Part IV* (*c*), L. Berman (*loc. cit.*), R. Stoy (*loc. cit.*) and H. Zanstra (*loc. cit.*).

TABLE IX (*b*)

Predicted Balmer Line Intensities *

T_ϵ	1000	5000	50000
H_α	26·1	28·2	30·8
H_β	10·0	10·0	10·0
H_γ	5·01	4·74	4·40

* Calculated by formula due to G. Cillié, *loc. cit.*

(*c*) *Line Intensities.*—It is possible to predict Balmer line intensity ratios from the velocity distribution of free electrons (i.e. T_ϵ), the capture coefficient, and the Einstein transition probabilities (computed from Quantum theory).* Theoretical intensities for different T_ϵ's are listed in

* F. G. Slack, *Phys. Rev.*, **31**, 527, 1928. Checked by Menzel and Pekeris, *loc. cit.*

Table IX (*b*) for comparison with observational values for four nebulæ listed in Table IX (*a*). There is a marked disagreement between observations of Balmer line intensities and predictions of intensities based on electron temperatures near 2000° K. In fact, the behaviour of the Balmer decrement is exactly contrary to that predicted by theory; it is *steeper* for *lower* T_ϵ.

After the discussion in the last section, it is probable that both the capture coefficient and the velocity distribution of the free electrons are accurately known, so that any failure of theory must be due either (*a*) to inaccuracy in the transition probabilities used, or (*b*) to self-reversal in the nebulæ.* With respect to (*a*), it has already been suggested by Ornstein and Burger,† after a test of the computed Einstein transition probabilities, that the weighting of substates in combining transition probabilities should be changed, more weight being given to the *P*-states. The effect of such a change on the theoretical Balmer line intensities is very small, and, in fact, in the direction contrary to observation. (Ornstein's weighting tends to *flatten* the decrement.)

With regard to (*b*), self-reversal (or the absorption of Balmer lines in the nebula), is possible only if there is a considerable number of *H* atoms in the two-quantum level. This can be brought about‡ by a very high radiation density of Lyman α line radiation ("Ly_α"), which is, indeed, predicted by Zanstra's theory of luminosity.§ Absorption will not decrease the total number of quanta in "H_α," the first Balmer line, since any Ly_β emitted will presumably be separated in its long path through the nebula to H_α and Ly_α. But since absorbed quanta of H_β and higher lines have the possibility of being re-emitted as H_α and Paschen lines, *the effect of self-reversal is to steepen the Balmer line decrement.* The amount of Ly_α radiation, and therefore the amount of steepening of the Balmer decrement due to self-reversal, increases with T_s, the temperature of the nuclear star.

It is now possible to offer an explanation of Table IX (*a*), though unfortunately the meagre data from only four nebulæ are not sufficient for a complete confirmation of the reasoning. Let us assume that the four nebulæ have approximately the same chemical composition; then, as T_s increases (from nebula to nebula) the energy density of Ly_α will increase, and increased self-reversal will cause a *steeper* Balmer decrement. Also, as T_s increases, the total amount of O^{++} and Ne^{++} (as formed by photo-ionization) in the nebula will increase, and therefore T_ϵ will decrease. But the flattening of the Balmer decrement due to this decrease in T_ϵ is small, as the theory predicts, and is entirely masked by the opposite effect due to self-reversal.

Part IV—The Visual Continuous Spectrum

(*a*) *Source.*—The strong intensity of continuous emission in the visual region between λ 3927 and λ 4800 needs explanation. From Tables IV,

* A suggestion believed due to Chandrasekhar.

† Ornstein and Burger, *Zs. f. Phys.*, **62**, 636, 1930.

‡ The metastability of the 2nd level is too small to account for appreciable self-reversal.

§ H. Zanstra, *Ap. J.*, **65**, 50, 1927.

V and VI it is seen to be over 50 per cent. of the maximum in the ultraviolet region—that is to say, of the same magnitude as the Ba_c itself. Possible sources of this continuous light are listed :

(1) Continuous emission due to captures on the Paschen level ($n=3$) of H (limit at λ 8210).
(2) H free-free transitions.
(3) He^+ continuous emission ($n=5$, limit at λ 5700).
(4) Continuous emission due to recombination of electrons with more highly charged ions.
(5) Ionization transfers.
(6) A continuous emission of H similar to that produced in the laboratory by Cohn.*
(7) Scattering.

We can now show that sources (1) to (6) are not capable of producing the observed intensity.

(1) and (2) are immediately rejected on the basis of equations (7) and (8), which show that the intensity so produced would be less than 0·0005 of the Ba_c at λ 3650 for T_ϵ near 2000° K. Even if T_ϵ were as high as 10,000° K, the contribution from these two sources is less than 0·05 of the Ba_c at λ 3650.

(3) Even if He^+ ions were as abundant as H atoms, equation (7), with T_ϵ near 2000° K, gives an intensity of less than 0·06 of the Ba_c at λ 3650. Moreover, this intensity would decrease rapidly (with $e^{-0\cdot8\times10^5/\lambda}$) from the above value at λ 4800 to 0·001 of the Ba_c at λ 3927.

The relative strength of possibility (4) suggested by Finkelnberg † relies on the fact that Z^4 appears in equation (7), but even for the abundance of other ions as high as $\frac{1}{100}$ that of hydrogen, the continuous emission so produced is only of the same magnitude as the He^+ emission in possibility (3) above.

(5) Inelastic collisions of the type ‡

$$He^+ + Ne \rightarrow He + Ne^+$$

probably contribute nothing to the continuous emission, since presumably they are similar to the process of sensitized fluorescence, in which the excess energy is completely converted into motion of the atoms,§ not continuous emission. In any case such a process is improbable because of the rarity of both reacting particles.

(6) Since there are no 25,000-volt electrons available in the nebula, continuous hydrogen emission as produced by Cohn in the laboratory cannot play an important part here.

From the foregoing discussion it would appear that nothing but scattering in the nebula is responsible for the high intensity of visual continuous

* W. M. Cohn, *Ast. Nach.*, **245**, 377, 1932.
† W. Finkelnberg, *Ap. J.*, **80**, 313, 1934.
‡ Suggested by Stoy, *P.A.S.P.*, **46**, 39, 1934.
§ Cario and Franck, *Zs. f. Phys.*, **17**, 202, 1923.

emission. The qualitative effects to be expected of pure scattering are simply the dimming of any star seen through the nebula, and the redistribution of nebular light and star light over the observable nebular disk. Further evidence for scattering as the origin of the visual continuous spectrum may then be derived from fig. 7, where the distribution of scattered (λ 4416) light across the nebular disk is compared with that of Balmer continuous (λ 3580) light. The scattering redistributes the nuclear star light fairly uniformly, in contrast to the fluorescent light (superposed on a

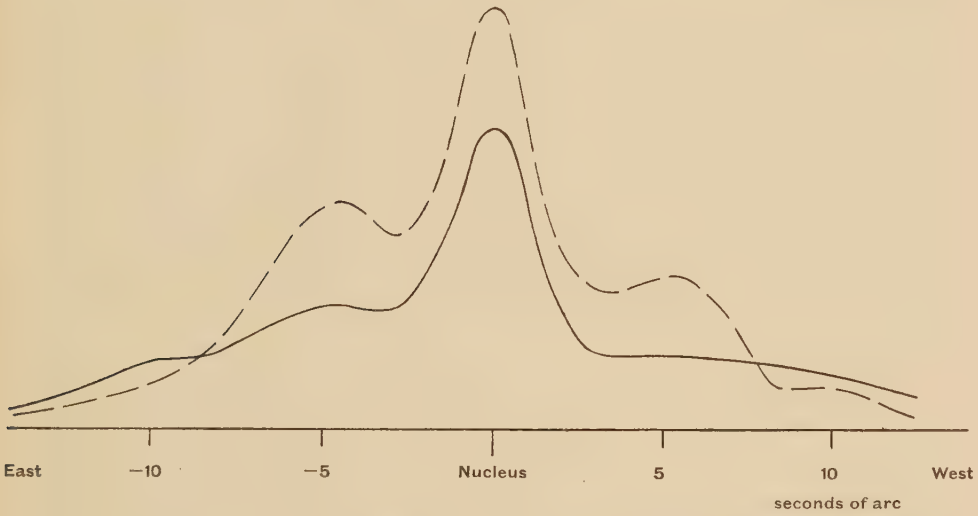


Fig. 7.—*Cross-sections of the continuous spectrum of N.G.C. 7662. Intensity as ordinate is plotted against distance from the nucleus as in figs. 3 and 5, and the curves represent averaged results from four plates, the corrections for unsteadiness of the image having been applied before averaging. The full line, a typical cross-section in the visual region, is for scattered light of wave-length λ 4416, and is to be compared with the dashed curve for fluorescent light of wave-length λ 3580, typical of the cross-sections in the* Ba_c.

background of scattered light), which increases rapidly to a marked maximum at about 5″ arc from the star.

(*b*) *Application of the Theory of Scattering and its Results.*—Having shown that the visual continuous emission is probably due to scattering, let us calculate quantitatively the consequences of pure scattering by means of a modification of Milne's theory of radiative equilibrium in nebulæ.* It is necessary to introduce into Milne's theory the fact that the particles may not be transferring all radiation of wave-length, λ, unaltered, but that they probably scatter (or reflect) only a fraction, μ, of the incident λ-radiation, and radiate the residual absorbed energy in the form of very long waves. [The black-body temperature in the nebula is between 10° K and 30° K.] This consideration leaves the equations † of transfer (for a wave-length λ in the visual region) as

* E. A. Milne, *Zs. f. Astrophys.*, **1**, 98, 1930.

† Milne's equations (3), (4), (5), p. 103, *loc. cit.*

$$\left.\begin{aligned}\cos\theta\frac{dI(\tau,\theta)}{d\tau}&=I(\tau,\theta)-B(\tau)\\ \cos\psi\frac{dI'(\tau,\psi)}{d\tau}&=B(\tau)-I'(\tau,\psi)\end{aligned}\right\} \quad (9)$$

and changes the equation of radiative equilibrium to

$$\mu\left[2\pi\int_0^{\pi/2}I(\tau,\theta)\sin\theta d\theta+2\pi\int_0^{\pi/2}I'(\tau,\psi)\sin\psi d\psi+\pi Se^{-(\tau_1-\tau)}\right]=4\pi B(\tau). \quad (10)$$

Here τ is optical depth $\left(\tau\equiv\int_0^r s\rho dr\right)$, I and I' the intensities of radiation outward and inward respectively, θ and ψ angles measured from the outward and inward direction respectively, and B the *Ergiebigkeit*, or light scattered into the beam.*

Milne gives four methods for measuring optical depth which are only approximate when used with ordinary photographs of planetary nebulæ, but which apply exactly to measurements of scattered light of a single wave-length in the visual continuous part of the spectrum. The simplest and most accurate method to use with these data is his third, represented by the equation †

$$\frac{\text{light from star}}{\text{scattered light from nebula}}\doteq\frac{Le^{-\tau_1}}{\pi\mu S(1-e^{-\tau_1})4\pi r_2{}^2}\doteq\frac{e^{-\tau_1}}{\mu(1-e^{-\tau_1})}. \quad (11)$$

The ratio on the left of equation (11) is measurable from the cross-sections of the visual continuous spectrum of N.G.C. 7662. The light from the star at λ 4416, for instance, is proportional to the area under the peak of the cross-section in fig. 7 cut off at the mean level of the nebular background. The remaining area is proportional to the scattered light from that part of the nebular disk which fell on the slit of the spectroscope. [The slit width of 0·5 mm. included 0·59 of the whole image.] In Table X the ratio of star light to total scattered light from the whole nebula is calculated for five different wave-lengths. Since no systematic change is noticeable, τ_1 is taken constant over the wave-lengths observed. The mean ratio from Table X is inserted in equation (11), and the τ_1's for four different values of μ are given in Table XI. Though they are all of reasonable magnitude, $\tau_1=2{\cdot}22$ for $\mu=0{\cdot}8$ has been used as a likely value in the following discussion.

The optical depths in N.G.C. 7662 and N.G.C. 7027 must be of about the same value. The only method of comparison is by means of the ratio of mean scattered intensity, V_c, to the maximum of the Ba_c. This ratio is 0·815 for N.G.C. 7662 ($\tau_1=2{\cdot}22$), 1·35 for N.G.C. 7027 (therefore $\tau_1>2{\cdot}22$?), and 0·54 for N.G.C. 6572 (where the star light is included; therefore $\tau_1<2{\cdot}2$).

* This analysis, applied to a different problem, has been fully treated by Chandrasekhar, *Zs. f. Astrophys.*, **9**, 266, 1935.

† The intensity of the scattered light escaping from the nebula is approximately proportional to μ, as may be seen from Chandrasekhar's results.

TABLE X

Computation of the Optical Depth of N.G.C. 7662

λ	Star Int.	Nebular Int. in Slit	Total Nebular Int.	Ratio Star to Nebula	Dev.
4416	1·03	3·49	5·93	0·174	+·035
4270	0·68	3·16	5·37	0·126	−·026
4152	0·86	3·39	5·76	0·149	−·004
4062	0·91	3·32	5·64	0·161	+·008
3927	0·98	3·73	6·34	0·154	+·001
			Mean =	0·153	±·010

TABLE XI

μ	e^{τ_1}	τ_1
(0)	(∞)	(∞)
0·3	22·8	3·12
0·5	14·0	2·64
0·8	**9·20**	**2·22**
1·0	7·53	2·02

(*c*) *The Effects of Scattering in the Gaseous Nebulæ.*—The dimming of a star's light seen through a scattering nebula of optical depth 2·2 is $e^{2\cdot2} = 9\cdot2$, or about $2^m\cdot4$. Now planetary nucleii as a class have lower absolute luminosities than ordinary stars of the same spectral type. To a great extent this anomaly has been explained by the white dwarf hypothesis, but there still remain cases of early type stars without white dwarf characteristics whose absolute luminosities are peculiarly low. A specific example is the nucleus of N.G.C. 2392 investigated by Cillié.* It is an ordinary O-type star (no Einstein red shift) of absolute magnitude $2^m\cdot9$, whereas the normal O-type absolute magnitude is less than $0^m\cdot0$. The anomaly may be explained by the fact that scattering takes place in the nebula and produces a dimming of $2^m\cdot9$.

The "temperature," T_s, of the nuclear star has been calculated, following Zanstra,† from the measured ratios of total nebular line intensities (from

TABLE XII

Intensities in the Pure Nuclear Star Spectrum of N.G.C. 7662

	λ 4800	λ 4416	λ 4270	λ 4152	λ 4062	λ 3927	λ 3680	λ 3630	λ 3580	λ 3530	λ 3480
I_s rel. to λ 3927	0·56	1·07	0·69	0·89	0·94	1·00	1·09	1·00	1·20	1·20	2·00
I_s rel. to total $H_\gamma = 10^4$	0·25	0·48	0·31	0·40	0·42	0·45	0·49	0·45	0·50	0·54	0·90
Acc. error	24%	37%	24%	24%	21%	57%	36%	21%	15%	36%	63%
Sys. error	15%	11%	5%	3%	2%	0%	0%	0%	1%	1%	1%
I_s for colour temp. 16000° K	0·58	0·75	0·82	0·89	0·93	1·00	1·16	1·19	1·23	1·26	1·30

* G. Cillié, *M.N.*, **94**, 48, 1933.

† H. Zanstra, *loc. cit.*, and *Pub. D.A.O.* (Victoria), **4**, 209, 1931.

Table VII) to stellar continuous spectrum (from Table XII). The results from four different methods, applied to N.G.C. 7662, are given in Table XIII,

TABLE XIII

Nuclear Temperature of N.G.C. 7662

Method of Determination	Direct	Corrected for Scattering
H recombination *	73000	45000
He^+ recombination *	(97000)	(72000)
"Nebulium" excitation *	67000	40000
Magnitude difference *	50000	(35000)
Line ratios †	32000	32000
Colour temperature	16000	16000
Probable value	...	40000

* H. Zanstra, *loc. cit.* † R. Stoy, *M.N.*, **93**, 588, 1933.

and though the "temperatures" are supposedly lower limits to the actual temperature, these lower limits are higher than ionization temperatures and the colour temperature derived from the nuclear spectrum in Table XII. Much of this anomaly is removed, however, when the effect of scattering in dimming the nuclear light is taken into account. In fact, the observed stellar luminosity is only 1/9·2 of the actual luminosity which should be adopted in Zanstra's theory, and the new values of T_s resulting from this correction are listed in the third column of Table XIII. It will be seen that the corrected values are in closer agreement with the commonly accepted ionization temperatures. Zanstra, and others who have closely followed his methods, have used for the star intensity only the blackening of a narrow central strip in the spectrum. Because nuclear light is scattered, however, they have under-estimated the intensity, and their determinations of "temperature" are consequently too high.*

(*d*) *The Nature of the Scattering Particles.*—Our discussion so far has been concerned firstly with establishing that the visual continuous spectrum is due to scattering, and secondly in discussing the effect of this scattering on the luminosity of the nuclear star and its "Zanstra-temperature." It is also of some interest, to see if anything can be discovered of the nature of the particles responsible for the scattering. It is clear at the outset that an optical depth of 2·2 cannot be produced by electron

* It should be noted that if the total nebular and nuclear (instead of pure nuclear) continuous spectrum has been used in Zanstra's methods, the error in T_s (due to neglected scattering) is partially compensated. The error then will be due only to μ being different from unity (*i.e.* due to absorption). Therefore Berman's "temperatures" of the N.G.C. 7027 nucleus (L. Berman, *loc. cit.*) based on nebular continuous spectrum only (since no star is visible) may be approximately correct.

scattering (where the free electrons are derived from the ionization of *H*) without assuming the enormous mass of 500 Suns for the nebula N.G.C. 7662. It is also clear, since the intensity of the visual continuous spectrum is independent of wave-length, that Rayleigh scattering plays no rôle, and that consequently particles must be of dimensions greater than two or three wave-lengths. This seems to leave only the possibility of scattering by meteoric dust particles having diameters of the order of 10,000 or 20,000 A. or greater.

Leaving the mean radius of the dust particles, q, as an unknown, we can find an expression for the mass of the nebula based on the measured optical depth, $\tau_1 = 2{\cdot}22$, and a material density of the particles, $p = 5$. By definition $\tau_1 = \int_{r_1}^{r_2} s\rho dr$ where r_1 and r_2 are the inner and outer radii of the nebula listed in Table XIV. s, the mass scattering coefficient, is defined as the scattering

TABLE XIV

Nebular Dimensions *

Nebula	Parallax	Outer Radius, r_2	Inner Radius, r_1
N.G.C. 7662	550 parsecs.	8,200 A.U.	4400 A.U.
N.G.C. 7027	1100 ,,	12,000 ,,	0?
N.G.C. 6572	760 ,,	11,000 ,,	0
I.C. 4997	1740 ,,	3,500 ,,	0?
I.C. 4593	1000 ,,	11,000 ,,	0

* From Vorentsov-Velyaminov, *Zs. f. Astrophys.*, **8**, 195, 1934.

(or reflecting) power per unit mass, or, since the effective area of each particle is πq^2 and the mass $(4/3)\pi pq^3$,

$$s = \frac{\mu\pi q^2}{(4/3)\pi pq^3} = \frac{3\mu}{4pq}.$$

Therefore

$$\tau_1 = \frac{3\mu}{4pq}\bar{\rho}(r_2 - r_1),$$

where $\bar{\rho}$, the mean of ρ, is the average mass of dust per cm.^{3} throughout the nebula. Solving

$$\bar{\rho} = \frac{4qp\tau_1}{3\mu(r_2 - r_1)} = 0{\cdot}3q\ 10^{-15}\ \text{gm./cm.}^3 \quad (q \text{ in cm.})$$

for N.G.C. 7662. The average number of dust particles per cm.^{3},

$$n = \frac{\bar{\rho}}{4/3\pi pq^3} = \frac{\tau_1}{\pi q^2\mu(r_2 - r_1)} = \frac{1{\cdot}6}{q^2}\ 10^{-17}/\text{cm.}^3$$

for N.G.C. 7662. The total mass of dust in the nebula,

$$M_d = \frac{4\pi}{3}\bar{\rho}(r_2^3 - r_1^3) = \frac{16\pi q p \tau_1}{9\mu}(r_2^2 + r_2 r_1 + r_1^2)$$
$$= q \times 10^3 \times \text{the Sun's mass (for N.G.C. 7662)}$$
$$= 1/10 \text{ Suns for } q = 10^{-4} \text{ cm.} = 10{,}000 \text{ A.}$$
$$= 10 \text{ Suns for } q = 10^{-2} \text{ cm.} = 0{\cdot}1 \text{ mm.}$$

It is probable, therefore, that the dimensions of the dust particles are between 10^{-4} and 10^{-2} cm.

It should be noted, however, that the dust hypothesis itself may encounter a difficulty when we compute the adsorption of nebular atoms on the solid particles. If every atom which strikes a dust particle is adsorbed or frozen on permanently, the density of the gas will decrease at a rate,

$$-\frac{dN_H}{dt} = 4\pi^2 n N_H q^2 \left(\frac{m_H}{2\pi k T_\epsilon}\right)^{3/2} \int_0^\infty v^3 e^{-\frac{1}{2}\frac{m_H v^2}{kT}}\, dv, \qquad (12)$$

where N_H is the number of atoms per cm.^{3} (the gas is assumed entirely H for simplicity), n is the number of dust particles per cm.^{3} from above, and T_ϵ is the electron (or hydrogen) temperature in the nebula. Equation (12) shows N_H to be an exponential function of the time, and, solving for the "half-life" of the nebula, we find

$$t_0 = \frac{\ln 2}{2q^2 n}\left(\frac{m_H}{2\pi k T_\epsilon}\right)^{1/2} = \frac{(\pi \ln 2)\mu(r_2 - r_1)}{2\tau_1}\left(\frac{m_H}{2\pi k T_\epsilon}\right)^{1/2}. \qquad (13)$$

Strangely enough, t_0 is independent of p and q for a given (measured) τ_1.* Substitution of the various constants in (13) produces the low value of $t_0 = 690$ years for N.G.C. 7662; *i.e.* this nebula would be half "swept up" by dust within itself in 690 years.

This difficulty can be avoided by supposing that a steady state has been reached in the nebula, where as many atoms are adsorbed from the gas by dust particles as are returned by some unspecified mechanism.†

As between free electrons, molecules or atoms, and dust particles as originators of the scattered continuous spectrum, there can be no doubt that the observational evidence favours dust particles. Too much stress should not, however, be put on such a speculative result. Our main conclusion is that the visual continuous spectrum is almost certainly produced by scattering, but the present data are hardly sufficient to determine completely the nature of the scattering particles.

Summary.—The following summary indicates the important results of this investigation as well as a general outline of the attack.

* Physically this merely means that the effective cross-sectional area of the dust has been taken the same for atomic collisions as for reflection of light.

† Such a mechanism might be band absorption of star light by ice formed on the particles. Fabry (*Ap. J.*, **40**, 241, 1914) has shown that such absorption can heat particles far above the black-body temperature; if they are heated to near 0° C., molecules will be evaporated off and returned to the gas.

Part I deals with the method of observation and reduction of spectral intensities in five planetary nebulæ.

Tables IV, V and VI in *Part II* list the continuous spectrum intensities and Table VII the line intensities resulting from the reduction. Figs. 3 and 5 illustrate intensity distributions across the nebular spectrum in continuous and line emission.

In *Part III* the Balmer continuous emission is shown to accord with the prediction that was based on the absorption coefficient law derived from quantum mechanics. The first measured electron temperatures, T_ϵ, are derived from the intensity distribution in this continuous emission (fig. 6 and Table IX(*a*)). They are low (near 2000° K), and not in agreement with observed Balmer line intensities, which also depend on the electron temperature. The cause of the discrepancy is shown to be self-reversal, which steepens the Balmer decrement. It is also noted that the measured electron temperatures are lower for nebulæ with stronger "nebulium" lines, and this is explained as the theoretically expected cooling effect of electron excitation.

Part IV is concerned with the strong continuous emission in the visual wave-lengths. It is first shown that the only probable source of such emission is scattering, and the optical depth of the scattering material in N.G.C. 7662 is measured from cross-sections of the visual continuous spectrum. The effects of scattering on observed luminosities and temperatures (*cf.* Table XIII) of planetary nuclei are deduced, and several anomalies thus explained. Since free electrons, atoms and molecules are none of them able to account for the scattering, we assume that it is due to meteoric dust, and calculations are therefore made of the space density and total mass of such dust in N.G.C. 7662 necessary to produce the observed optical depth.

It is a pleasure to express my deepest thanks to Professor H. H. Plaskett, who has supplied the valuable observational material for this investigation, made many helpful suggestions, and taken great interest in discussing the results. I am also indebted to Professor Milne for criticism of the theoretical section, and to Sir Montagu Pollock for reading the manuscript and proof.

University Observatory,
Oxford :
April 1936.

ON THE GALACTIC ROTATION OF THE GLOBULAR CLUSTERS: SOME COMMENTS ON DR. MINEUR'S PAPER.

Frank K. Edmondson.

(Communicated by Prof. H. N. Russell)

Dr. Mineur recently published a paper * in which he analysed the 21 radial velocities compiled by S. W. Shiveshwarkar.† At nearly the same time my own treatment of the problem based on 26 velocities (including unpublished Lowell and Mount Wilson material) was printed.‡ In view of the different conclusions reached in these two papers, it seems desirable to examine the hypotheses used by Dr. Mineur and myself as well as to compare the numerical results obtained by each of us.

In addition to assuming that the law of velocity in the galaxy is of the form

$$V = ar + br^2 + crz^2,$$

Dr. Mineur makes the following fundamental assumptions at the beginning of his analysis :—

1. The orbits are circular.
2. The clusters revolve around the axis of the galaxy (*i.e.* parallel to the galactic plane regardless of the value of z).
3. The direction of rotation is the same for all clusters.

With regard to the first of these assumptions, figure 3 in *Lowell Observatory Bulletin* 88 shows that the hypothesis of circular orbits is a reasonable one. The argument is, in short, that for the clusters studied the largest observed radial velocities (corrected for solar motion) occur when the angle *Sun-Cluster-Centre* is nearly 90°. For example, consider these conspicuous cases (Vel. > 200 km./sec.) :

N.G.C.	sin θ	θ	Vel.	V_r
6218	0·90	65°	+236	540
6333	·75 :	50° :	+256	385 :
6402	·96	75°	+259	940
6779	·95	70°	+225	730

If we assume that the motions are predominantly radial, the velocities in the last column indicate that these clusters are escaping from the galaxy (with the possible exception of N.G.C. 6333).

* *M.N.*, 96, 61, 1935.
† *M.N.*, 95, 555, 1935.
‡ *A.J.*, 1036. Reprinted as *Lowell Observatory Bulletin* 88, 1935.

The second of Dr. Mineur's assumptions seems to be less reasonable. In the form in which it is stated, it represents a condition which is dynamically impossible, for a cluster with (let us say) $r = 5$ kpc. and $z = 10$ kpc. cannot revolve around the *axis* of the galaxy. The assumption can be reconciled with the laws of dynamics by interpreting it to mean that each cluster has at this moment reached its maximum distance from the galactic plane (*i.e.* the orbits are inclined, but each cluster is exactly 90° away from the node). It seems *a priori* highly improbable that this hypothesis is true. In addition, there is direct evidence against it in Tables IV and V of *Lowell Observatory Bulletin* 88. The argument can be summarized as follows: The angle (ϕ) that the parallel motion makes with the direction of maximum possible radial velocity can be computed from the co-ordinates of the cluster, and the angle (ψ) that the real motion of the cluster makes with the direction of maximum possible radial velocity can be computed (with an ambiguity of sign) from the observed radial velocity. It is necessary that $\phi = \psi$ if the orientation of the velocity is to be parallel to the galactic plane, but an observed equality of these angles is not sufficient to prove that the motion is parallel to the galactic plane, since the orientation might be $\phi + \psi$ instead of zero. The observed frequency distribution of $|\phi - \psi|$ (*i.e.* the minimum possible deviation from parallel motion) is summarized in the following table:—

0°–30°	30°–60°	60°–90°
19	5	2

Even on the basis of the most favourable assumption (namely, that the orientation is $|\phi - \psi|$ in every case), it is apparent that the instantaneous velocities are not parallel to the galactic plane to a sufficient degree of approximation to justify Dr. Mineur's second assumption, and its failure is reflected in the residuals in his Table I. The outstanding example is N.G.C. 5024 (M53), a cluster only 10° from the galactic pole with an observed radial velocity (corrected for solar motion) of −186 km./sec. Obviously the inclination of this orbit is nearly 90°.

Dr. Mineur's third assumption is also contradicted by some of the observed radial velocities. N.G.C. 1851, for example, has a radial velocity (corrected for solar motion) of +71 km./sec., and for either of the two possible inclinations the orbit is retrograde. N.G.C. 6440 should have a positive radial velocity if the orbit is direct, and the observed value is −132 km./sec.

It should also be pointed out that assumptions 2 and 3 are not independent from an observational standpoint. If we assume that 2 is approximately true, then the observed radial velocities require that the orbits be nearly evenly divided between direct and retrograde. On the other hand, if we assume that 3 is true (in all possible cases), then the observed radial velocities require very large deviations from 2. In other words, Dr. Mineur's

second and third assumptions cannot be true simultaneously. Assuming 3 to be true, the observed frequency distribution of $|\phi \pm \psi|$ is given in the following table :—

0°–30°	30°–60°	60°–90°
13	6	7

It is of interest to compare the inclinations of the orbit planes which follow from these various hypotheses. Table I gives the frequency distribution of the inclinations for the following cases : (1) Mineur's assumption that the motions are instantaneously parallel to the galactic plane ; (2) the distribution adopted in *Lowell Observatory Bulletin* 88 (corresponding to even division between direct and retrograde motions) ; (3) direct orbits in all cases for which this is possible ; and (4) hypothetical distribution corresponding to a random distribution of the poles of the orbits. Pearson's χ^2 test indicates that (2) differs significantly from a random distribution. On the other hand, there is no significant difference between (3) and a random distribution, the probability of a less probable sample being 0·25.

Table I

0°–30°	30°–60°	60°–90°	Case
16	6	4	(1)
9	12	5	(2)
7	9	10	(3)
3	10	13	(4)

Unfortunately, Mineur's numerical results are affected by his failure to correct Shapley's distances for the absorption of light in space. Hence, a detailed discussion of the individual residuals from his analysis will not be attempted, although it is of interest to compare the distribution of residuals in his Table I with the distribution of residuals obtained by subtracting only the effect of the Sun's orbital motion of 300 km./sec. toward longitude 57°. Fig. 1 shows that while the reduction of residuals obtained by Dr. Mineur is probably significant (the standard deviation is reduced by 30 per cent.), it is disappointingly small when we recall that the rotation term was introduced to account for the large residuals from the solar motion. It should be pointed out that some reduction of residuals is to be expected, since the preceding discussion indicates that some (but not all) of the clusters are moving in the manner required by Dr. Mineur's theory.

Dr. Mineur states that the purpose of his empirical velocity formula is to find out the importance of V and its variations, and he feels that his results are reasonable because of the similarity of the velocity of a cluster

in the neighbourhood of the Sun (− 310 km./sec.) with that of the rotation of the stars in our neighbourhood. Apparently Dr. Mineur believes that if there is a general law connecting velocity and distance in the galaxy, then

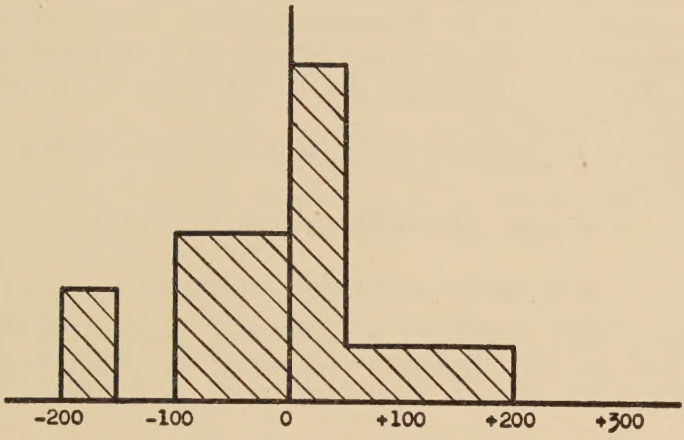


FIG. 1.

it is not unreasonable to suppose that it will reduce to the Sun's velocity upon substitution of the Sun's co-ordinates. This suggests a direct method of evaluating a, X and Y from the clusters actually used in the rotation analysis, instead of assuming that X and Y are known. For it follows that

$$\sqrt{X^2+Y^2}=V_\odot=a+b$$

upon substitution of $r=1$, $z=0$ in the formula for V. Since b is known, this extra equation of condition enables us to separate a from X and Y. Performing the calculation, we find :

$$a=-145, \qquad X=-90, \qquad Y=+24.$$

According to this result, V is small between 0 and 6 kiloparsecs from the galactic centre and becomes rapidly larger with increasing distance. The velocity is about $+1500$ km./sec. at a distance of 25 kiloparsecs from the galactic centre. This unexpected result, obtained by assuming the truth of the hypothesis used by Dr. Mineur to check his theory, is inconsistent with his statement that the agreement between his formula and the Sun's velocity in the galactic rotation is not a consequence of his method of determining the constant a.*

It remains to be explained how it is possible for two such different velocity laws as

$$V = 300 \quad (\textit{Lowell Observatory Bulletin } 88)$$

and

$$V = -550r + 240r^2 + 280rz^2 \quad \text{(Mineur)}$$

to reproduce Oort's coefficients of differential galactic rotation. The reason is that the only dynamical information we obtain from Oort's coefficients is

(1) $$A - B = \frac{V}{R} \qquad \text{(at the Sun)},$$

(2) $$A + B = -\frac{dV}{dR} \quad \text{(at the Sun)},$$

and the observed values of A and B are such as to make

$$\frac{V}{R} = \frac{300 \text{ km./sec.}}{10{,}000 \text{ parsecs}}, \quad \text{and} \quad \frac{dV}{dR} \sim 0.$$

Hence, any velocity law which gives $V = 300$ and $\frac{dV}{dR} \sim 0$, when $r = 10$ kpc. and $z = 0$, will reproduce Oort's coefficients.† Since the maximum of Mineur's velocity curve occurs near $r = 10kpc$ and amounts to nearly 300 km./sec., he finds agreement with Oort's coefficients. The preceding discussion indicates that we must attribute the agreement he finds to his method of evaluating the constant a.

An interpolation formula of the form

$$V = ar + br^2 + crz^2$$

is too inflexible to be generally useful, chiefly because a constant term is lacking. It gives $V = 0$ at $r_0 = 0$, and $r_1 = -(cz^2 + a)/b$ (Mineur takes these values to be the range of validity of the formula), has a maximum at $r_{max} = r_1/2$, and is symmetrical about this maximum. Hence, it cannot represent a rapid increase of V for small distances and a slow decrease for large distances. The lack of adaptability of the formula is obvious if we consider that an

* If X and Y are known, the observed radial velocities should be corrected before the coefficients of the rotation term are determined.

† That is why Oort's coefficients alone are not sufficient to determine uniquely the distribution of mass in the galaxy.

observer living in a hypothetical galaxy with V = constant (and assumptions 1, 2 and 3 all fulfilled exactly) would not be able to discover the true velocity law with this formula, but would arrive at fictitious results.

Summary

1. The observed radial velocities of the globular clusters indicate that (if the orbits are circular) either : (1) The motions are evenly divided between direct and retrograde, and the inclinations deviate from a random distribution in the sense that small inclinations are favoured ; or (2) the motions are nearly all direct, and the inclinations do not deviate significantly from a random distribution. Neither of these conclusions is in accord with Dr. Mineur's hypotheses.

2. It is shown that Dr. Mineur's equations do not represent the observed radial velocities of the globular clusters with sufficient accuracy.

3. The agreement between Dr. Mineur's formula for V and Oort's coefficients appears to be due to the method that was used to evaluate the constant a.

4. The formula $V = ar + br^2 + crz^2$ is not a satisfactory interpolation formula because of its lack of flexibility.

Harvard College Observatory :
1936 *March* 15.

DESCRIPTION OF CHART OF THE MILKY WAY DEPOSITED WITH THE SOCIETY.

Otto Boeddicker, Phil.D.

The general view of the Milky Way which I have deposited with the Royal Astronomical Society is the very first combination of the sectional parts taken directly from the sky. It was done at Birr Castle Observatory in the year 1889, the whole work being spread over the five years 1884 to 1889. As it had lost some of its freshness by the work of drawing and repeated comparing with the heavens, as well as by its having passed through the hands of various astronomers interested in the undertaking, I copied it once only on the same scale. This copy remained at Birr Castle. The present original drawing, somewhat dilapidated and torn, remained in my possession. When I left Ireland in the beginning of 1915 February I had to leave it behind in Birr, packed in a box along with a few of my household goods, and I did not see it again until seven years later. Having passed through many vicissitudes like myself, it was not really unrolled again until two years ago.

My Milky Way drawing was published by Messrs. Longmans in 1892, lithographed by Mr. Wesley of this Society, a most skilful and eminent draughtsman and lithographer. Photographic reproductions had proved

to be impossible forty-five years ago. In 1935, however, I thought it worth while to try what modern photography could do with the assistance of a most experienced photograper, Mr. Albert Kuban of Staufen im Breisgau, Baden. The result was a bitter disappointment. The drawing-paper was found to have deteriorated to such an extent that the negative was covered with most irregular and—as regards the delicate detail of the drawing—destructive patches, streaks and blotches. So I was obliged to abandon all ideas of a photographic reproduction.

Thus, being in my eighty-third year and, not unnaturally, anxious to find a permanent home for the drawing, I asked my friend the Hon. G. L. Parsons, the surviving son of the fourth Earl of Rosse, to inquire whether the Royal Astronomical Society would perhaps be prepared to take it into its care. The Society acceded to this request.

It is indeed a great satisfaction to me to be able to place my drawing in such safe and honourable keeping, and I herewith express my warmest thanks to the Royal Astronomical Society for granting it.

Freiburg i. Breisgau,
Baden, Germany,
54 *Zasiusstrasse :*
1936 *March* 13.

Errata

M.N., **96**, 179, 1936, E. A. Milne, "The Pressure in the Interior of a Star."

Dr. Lloyd Motz, of the Department of Astronomy, Columbia University, has kindly pointed out to me that the extreme left-hand members of inequalities (8) and (12) are algebraically incorrect. The corrected inequalities are

$$\nu\pi(P_c - P_1)r_1^{4-\nu} + \frac{4-\nu}{8}\frac{GM^2}{r_1^{\nu}} > I_\nu > \frac{GM^2}{2r_1^{\nu}}, \tag{8}$$

and

$$\pi(P_c - P_1)r_1^3 + \frac{3}{8}\frac{GM^2}{r_1} > \Omega > \frac{GM^2}{2r_1}. \tag{12}$$

E. A. M.

M.N., **96**, 426, 1936 (paper by Messrs. Greaves and Martin), line 3 from top of page, *for* multiplied *read* divided. (The computations referred to were performed correctly, but a slip was made in writing the text of the paper.)